The value distribution of zeta and L-functions

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More interesting questions on the critical line:
What do typical values of $\zeta(1/2 + it)$ look like?
Answer: Selberg's central limit theorem. Typical values are either big or small, and not usually of constant size.

Larger values.

What can say about the moments

$$\int_0^T |\zeta(1/2+it)|^{2k} dt?$$

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How large/small can $|\zeta(\sigma+it)|$ be? Special interest: $\sigma = 1/2$.

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How large/small can $|\zeta(\sigma+it)|$ be? Special interest: $\sigma = 1/2$.
Lindelöf Hypothesis. Gulf between bounds on RH and known Ω–results.

Fyodorov–Keating conjectures:

For t chosen randomly from $[T, 2T]$, distribution of

$$\max_{t \le u \le t+1} |\zeta(1/2 + iu)|?$$

$$\int_t^{t+1} |\zeta(1/2 + iu)|^{2k} du?$$

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Conjectured answers based on Branching Brownian Motion.

Known for analogous quantity in random matrix theory.

Partial crude progress for $\zeta(s)$.

Links to behavior of random multiplicative functions.

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d a fundamental discriminant with $|d| \le X$.
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3. Families arising from modular forms. Orthogonal example.
E an elliptic curve over $\mathbb{Q}$ of conductor N.
E_d — quadratic twist by fundamental discriminant d
Behavior of central value $L(1/2, E_d)$ (assume root number 1).

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These families often yield arithmetic information.
New features/problems: could happen that many central values are zero, although not expected to be the case.

Distribution of values at the edge of the critical strip

Family of quadratic Dirichlet characters χ_d; d a fundamental discriminant.

Recall $(\frac{d}{\cdot})$ primitive Dirichlet character $\pmod{|d|}$. Built out of

$$\Big(\frac{-4}{\cdot}\Big), \quad \Big(\frac{8}{\cdot}\Big), \quad \Big(\frac{-8}{\cdot}\Big), \quad \Big(\frac{(-1)^{(p-1)/2}p}{\cdot}\Big).$$

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Almost periodicity.

For example: if d_1 and d_2 are fundamental discriminants with $d_1 \equiv d_2 \pmod{840}$ then

$$L(2, \chi_{d_1}) \approx L(2, \chi_{d_2}).$$

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Indeed $840 = 8 \times 3 \times 5 \times 7$ and so

$$|L(2, \chi_{d_1}) - L(2, \chi_{d_2})| \leq \sum_{\substack{\exists p \geq 11 \\ p|n}} \frac{1}{n^2} < 0.06$$

Analogous result at the edge of the critical strip. No longer true always, but most of the time

$$L(1, \chi_{d_1}) \approx L(1, \chi_{d_2})$$

if $d_1 \equiv d_2 \pmod{4 \prod_{p \le z} p}$.
That is, apart from a small number of exceptional $|d| \le X$,

$$L(1, \chi_d) \approx \prod_{p \le z} \left(1 - \frac{\chi_d(p)}{p}\right)^{-1}.$$

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Probabilistic model to analyze the truncated product.
Independent random variables defined on primes:

$$X(p) = \begin{cases} 1 \text{ or } -1 & \text{with probability } \frac{p}{2(p+1)} \\ 0 & \text{with small probability } \frac{1}{p+1}. \end{cases}$$

Random Euler product:

$$L(1,\mathbb{X}) = \prod_{p}\left(1-\frac{X(p)}{p}\right)^{-1}.$$

A qualitative argument

Partial summation and Pólya–Vinogradov:

$$
\begin{aligned}
L(1,\chi_d) &= \sum_{n\le N} \frac{\chi_d(n)}{n} + \int_N^\infty \sum_{N<n\le y} \chi_d(n) \frac{dy}{y^2} \\
&= \sum_{n\le N} \frac{\chi_d(n)}{n} + O\Big(\frac{\sqrt{|d|}\log|d|}{N} \Big).
\end{aligned}
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$$= \sum_{n\le N} \frac{\chi_d(n)}{n} + O\Big(\frac{\sqrt{|d|}\log|d|}{N}\Big).$$

Take $N = \sqrt{X}(\log X)^{100}$, and z tending to infinity slowly with X.

$$L(1,\chi_d) \approx \sum_{\substack{n\le N\\ n\in\mathcal{S}(z)}} \frac{\chi_d(n)}{n} + \sum_{\substack{n\le N\\ n\notin\mathcal{S}(z)}} \frac{\chi_d(n)}{n}$$

$\mathcal{S}(z)$ — smooth numbers, $p|n \implies p\le z$.
First sum depends only on primes $p\le z$: Fixing d in a progression $(\bmod\ 4\prod_{p\le z} p)$ fixes this sum. Can handle $z \le \frac{1}{10}\log X$
Second sum is small in mean square as $z\to\infty$.

For any choice of signs ϵ_p for $p \leq \frac{1}{10} \log X$

$$\#\{|d| \leq X : \quad \chi_d(p) = \epsilon_p \text{ for all } p \leq z\} \sim \frac{6}{\pi^2} X \prod_{p \leq z} \Big(\frac{p}{2(p+1)} \Big).$$

Therefore

$$\sum_{\substack{n \leq N \\ n \in \mathcal{S}(z)}} \frac{\chi_d(n)}{n} \approx \prod_{p \leq z} \Big(1 - \frac{\chi_d(p)}{p} \Big)^{-1}$$

behaves like random Euler product

$$\prod_{p \leq z} \Big(1 - \frac{\mathbb{X}(p)}{p} \Big)^{-1} \approx L(1, \mathbb{X}).$$

Second term in mean square:

$$\sum_{|d|\le X}^{\flat}\Big(\sum_{\substack{n\le N\\ n\notin\mathcal{S}(z)}}\frac{\chi_d(n)}{n}\Big)^2=\sum_{\substack{n_1,n_2\le N\\ n_1,n_2\notin\mathcal{S}(z)}}\frac{1}{n_1n_2}\sum_{|d|\le X}^{\flat}\Big(\frac{d}{n_1n_2}\Big).$$

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Sum over d is a character sum over $\left(\frac{\cdot}{n_1n_2}\right)$. If $n_1n_2\ne\square$ this cancels, by Pólya–Vinogradov.

Nuisance: d must be (essentially) square-free:

$$\begin{aligned}\sum_{|d|\le X}\Big(\sum_{k^2|d}\mu(k)\Big)\Big(\frac{d}{n_1n_2}\Big)&=\sum_{k\le\sqrt{X}}\mu(k)\sum_{\substack{|d|\le X\\ k^2|d}}\Big(\frac{d}{n_1n_2}\Big)\\&\ll\sum_{k\le\sqrt{X}}\min\Big(\sqrt{n_1n_2}\log X,\frac{X}{k^2}\Big)\\&\ll X^{\frac34}(\log X)^2.\end{aligned}$$

Contribution of the terms $n_1 n_2 \neq \square$ ("non diagonal" / "off diagonal"):

$$\ll \sum_{n_1, n_2 \leq N} \frac{1}{n_1 n_2} X^{\frac{3}{4}} (\log X)^2 \ll X^{\frac{3}{4}} (\log X)^4.$$

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Main terms – diagonal contribution –

$$\sim \sum_{\substack{n_1,n_2 \le N \\ n_1 n_2 = \square \\ n_1,n_2 \notin \mathcal{S}(z)}} \frac{1}{n_1 n_2} \frac{6}{\pi^2} X \prod_{p \mid n_1 n_2} \left(\frac{p}{p+1} \right).$$

$n_1 = a m_1^2, \quad n_2 = a m_2^2$. At least one of a, m_1, or m_2 must be $> z$.

$$\text{Main term } \ll X \sum_{a,m_1,m_2} \frac{1}{a^2 m_1^2 m_2^2} \ll \frac{X}{z}.$$

Conclude: For most d, the second term (over non-smooth n) is small.

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Toy problem: $X(n) = \pm 1$ independently for each n. Consider

$$\sum_{n=1}^{\infty} \frac{X(n)}{n}.$$

Heavily determined by first few values of $X(n)$.

Probability that $\sum_{n=1}^{\infty} X(n)/n > 10$?

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Random Euler product distribution:

$$\mathrm{Prob}\Big(L(1,\mathbb{X}) \geq e^{\gamma}\tau\Big) = \exp\Big(-\frac{e^{\tau - C_1}}{\tau} + O\Big(\frac{e^{\tau}}{\tau^2}\Big)\Big)$$

$$C_1 = \int_0^1 \tanh y \frac{dy}{y} + \int_1^{\infty} (\tanh y - 1)\frac{dy}{y} = 0.8187\ldots.$$

Similarly for $\mathrm{Prob}(L(1,\mathbb{X}) \leq \zeta(2)/(e^{\gamma}\tau))$.
Moral: $L(1,\chi_d) \in [1/10, 10]$ usually!

Quantitative/Uniform results?

How well does the probabilistic model approximate the distribution of $L(1, \chi_d)$?

Range of values τ for which

$$\#\{|d| \le X : L(1, \chi_d) \ge e^{\gamma}\tau\} \approx \#\{d\}\mathrm{Prob}(L(1, \mathbb{X}) \ge e^{\gamma}\tau).$$

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Largest viable range for uniformity?

$$\tau \leq \tau_{\max} = \log_2 X + \log_3 X + C_1 + o(1).$$

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Suggests the conjectures:

$$\max_{|d| \leq X} L(1, \chi_d) = e^{\gamma}(\log_2 X + \log_3 X + C_1 + o(1))$$

$$\min_{|d| \leq X} L(1, \chi_d) = \zeta(2)\Big(e^{\gamma}(\log_2 X + \log_3 X + C_1 + o(1))\Big)^{-1}.$$

Approximating $L(1, \chi_d)$ by short Euler products

Generalized Riemann Hypothesis: (smooth weighting)

$$\sum_n \Lambda(n)\chi(n)e^{-n/x} \ll \sqrt{x}\log(qx).$$

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Guarantees that $\chi(p)$ cancel out, once $p > (\log q)^2$.

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Probabilistic model suggests that the least quadratic non-residue is $\ll (\log p)\log\log p$.

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On GRH

$$L(1,\chi_d) \approx \prod_{p\le(\log|d|)^2}\left(1-\frac{\chi_d(p)}{p}\right)^{-1}.$$

Probabilistic model suggests enough to take product up to $(\log|d|)$.

On GRH

$$\zeta(2)((2+o(1))e^{\gamma}\log\log|d|)^{-1} \le L(1,\chi_d) \le (2+o(1))e^{\gamma}\log\log|d|$$

Unconditional bounds:

$$C(\epsilon)|d|^{-\epsilon} \le L(1, \chi_d) \le \frac{1}{4}\Big(2 - \frac{2}{\sqrt{e}} + o(1)\Big) \log |d|.$$

Lower bound: Siegel's (ineffective) theorem.
Upper bound: follows from Burgess & refinement by Stephens.
Related to least quadratic non-residue being $\le p^{1/(4\sqrt{e})+\epsilon}$.

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Related to least quadratic non-residue being $\leq p^{1/(4\sqrt{e})+\epsilon}$.
Much better results if a small number of bad characters are omitted.
Large sieve/zero density estimates.
Apart from at most $Q^{2/A+\epsilon}$ primitive characters $\chi \pmod q$ with $q \leq Q$

$$L(1,\chi) \approx \prod_{p \leq (\log q)^A} \Big(1 - \frac{\chi(p)}{p}\Big)^{-1}.$$

Can use this to compute moments of $L(1,\chi)$ and compare with moments of random Euler products.

Uniform results

Theorem: (Granville & S.) Uniformly in $|z| \le \log X/(500(\log_2 X)^2)$

$$\sideset{}{^\flat}\sum_{\substack{|d|\le X\\ d\notin\mathcal{L}}} L(1,\chi_d)^z = \frac{6}{\pi^2} X \mathbb{E}(L(1,\mathbb{X})^z) + O\Big(X \exp\Big(-\frac{\log X}{5\log_2 X}\Big)\Big),$$

where $\mathcal{L}$ excludes $\ll \log X$ discriminants having a possible Landau–Siegel zero.

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where $\mathcal{L}$ excludes $\ll \log X$ discriminants having a possible Landau–Siegel zero.

Theorem: (Granville & S.) $e \le A \le \log_2 x$.

$$\#\{|d|\le X:\ L(1,\chi_d)\ge e^\gamma\tau\} = \#\{d\}\exp\Big(-\frac{e^{\tau-C_1}}{\tau}\Big(1+O\Big(\frac{1}{A}+\frac{1}{\tau}\Big)\Big)\Big),$$

uniformly in the range

- $\tau \le \log_2 X + \log_4 X - \log_2 A - 20.$
- $\tau \le \log_2 X + \log_3 X - \log_2 A - 20$ on GRH.

Similar result for small values of $L(1,\chi_d)$.

Distills/refines work of many: Chowla, Elliott, Montgomery & Vaughan, Heath-Brown, Graham & Ringrose.
The range $\tau \le \log_2 X + \log_4 X - \log_2 A - 20$ established a strong form of a conjecture of Montgomery & Vaughan. Proportion of $|d| \le X$ with $L(1, \chi_d) \ge e^\gamma \log_2 X$ is

$$\exp\Big(- (e^{-C_1} + o(1)) \frac{\log X}{\log \log X} \Big).$$

Key ingredient: Work of Graham & Ringrose on character sums.
The least quadratic non-residue can be as large as $\log p \log_3 p$.

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Key ingredient: Work of Graham & Ringrose on character sums.
The least quadratic non-residue can be as large as $\log p \log_3 p$.
The GRH range almost obtained a second conjecture of Montgomery & Vaughan. Gives extreme values of size $e^\gamma(\log_2 X + \log_3 X - \log(2\log 2) - \epsilon)$. Proportion of d for which this happens is $\gg X^{-\frac{1}{2}}$.
Motivated our conjectures on extreme values for $L(1, \chi_d)$.