Moments of zeta and L-functions

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May 22, 2019

Moments of $\zeta(s)$

Classical problem: Asymptotics for

$$M_k(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} dt.$$

Hardy-Littlewood:

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^2 dt \sim T \log T.$$

Ingham:

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^4 dt \sim \frac{1}{2\pi^2} T(\log T)^4.$$

No other moments are known.

RH implies Lindelöf Hypothesis – $|\zeta(\frac{1}{2} + it)| \ll (1 + |t|)^{\epsilon}$ – which is equivalent to

$$M_k(T) \leq C(k, \epsilon) T^{1+\epsilon}.$$

Conrey-Ghosh(-Gonek) conjecture:

$$M_k(T) \sim a_k g_k T(\log T)^{k^2},$$

with

$$a_k = \frac{1}{(k^2)!} \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \left(\sum_{a=0}^{\infty} \frac{1}{p^a} \binom{k+a-1}{a}^2 \right),$$

and

$$g_1 = 1, \quad g_2 = 2, \quad g_3 = 42, \quad g_4 = 24024, \quad g_k = ???$$

Keating–Snaith Conjectures:

$$g_k = (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}.$$

Questions

Where do these conjectures come from?
Why is the rate of growth $(\log T)^{k^2}$?
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Related questions for families of L–functions:

$$\sum_{\chi \pmod q} |L(\tfrac{1}{2},\chi)|^{2k} \sim C_k q(\log q)^{k^2}$$

$$\sum_{|d|\le X} L(\tfrac{1}{2},\chi_d)^k \sim C_k X(\log X)^{k(k+1)/2}$$

$$\sum_{|d|\le X} L(\tfrac{1}{2},E\times\chi_d)^k \sim C_k X(\log X)^{k(k-1)/2}$$

Analogous problems over function fields.

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Analogous problems over function fields.
Progress toward these conjectures? Asymptotics known in some cases. Lower bounds of the right order? Upper bounds of the right order?

Heuristic 1 – Diagonal contributions of Dirichlet series

$$\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}$$

Ignoring convergence & focussing just on diagonal terms:

$$\int_T^{2T} |\zeta(\sigma+it)|^{2k} dt = \sum_{m,n} \frac{d_k(m)d_k(n)}{(mn)^\sigma} \int_T^{2T} \left(\frac{m}{n}\right)^{it} dt = T \sum_n \frac{d_k(n)^2}{n^{2\sigma}}.$$

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True if $\sigma \geq 1$.
Answer converges if $\sigma > 1/2$, and correct if Lindelöf Hypothesis holds.
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$$T \sum_{n \leq T} \frac{d_k(n)^2}{n}.$$

$$\sum_{n=1}^{\infty} \frac{d_k(n)^2}{n^s} = \zeta(s)^{k^2} A_k(s)$$

$$A_k(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{k^2} \left(1 + \frac{d_k(p)^2}{p^s} + \dots\right)$$

$$\sum_{n \le T} \frac{d_k(n)^2}{n} \sim \mathrm{Res}_{s=0}\left(\zeta(s+1)^{k^2} A_k(s+1) \frac{T^s}{s}\right) \sim a_k (\log T)^{k^2}.$$

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Similar calculations in other families:

$$\sideset{}{^\flat}\sum_{|d| \le X} L(\tfrac{1}{2}, \chi_d)^k \longleftrightarrow \#\{|d| \le X\} \sum_{n \le X} \frac{d_k(n^2)}{n} \sim C_k X (\log X)^{k(k+1)/2}.$$

Note $d_k(p^2) = k(k+1)/2$, which determines the power of $\log X$.

Quadratic twists of an elliptic curve E

$$L(s, E_d)^k = \sum_{n=1}^{\infty} \frac{a_k(n)}{n^s} \chi_d(n)$$

$a_k(n)$ multiplicative function:

$$\left(1 - \frac{\alpha_p}{p^s}\right)^{-k} \left(1 - \frac{\beta_p}{p^s}\right)^{-k} = \sum_{\ell=0}^{\infty} \frac{a_k(p^\ell)}{p^{\ell s}}$$

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Expect

$$\sum_{\substack{|d| \le X \\ d \in \mathcal{E}}}^{\flat} L(\tfrac{1}{2}, E_d)^k \longleftrightarrow \#\{d\} \sum_{n \le X} \frac{a_k(n^2)}{n}$$

Power of $\log X$ determined by average value of $a_k(p^2)$.

$$\begin{aligned} a_k(p^2) &= (k\alpha_p)(k\beta_p) + \frac{k(k+1)}{2}(\alpha_p^2 + \beta_p^2) \\ &= \frac{k(k-1)}{2} + \frac{k(k+1)}{2}(1 + \alpha_p^2 + \beta_p^2). \end{aligned}$$

Heuristic 2: Extrapolations of Selberg's theorem

An analogy with the divisor function

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Better version: $\omega(n)$ is approximately Poisson with parameter $\log\log N$:

$$\#\{n \leq N : \quad \omega(n) = k\} \sim N \frac{(\log\log N)^{k-1}}{(k-1)!} e^{-\log\log N}.$$

True as stated for fixed k; uniformity in k – work of Sathe–Selberg.

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X is a Poisson random variable with parameter λ. Then

$$\mathbb{E}(t^X) = e^{-\lambda} \sum_{\ell=0}^{\infty} \frac{t^\ell \lambda^\ell}{\ell!} = e^{\lambda(t-1)}.$$

Dominated by terms $\ell \approx t\lambda$.

Since $d_k(n)$ looks like $k^{\omega(n)}$ this suggests

$$\frac{1}{x}\sum_{n\le x} d_k(n) \asymp \exp((k-1)\log\log x) = (\log x)^{k-1}.$$

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Note that the constant in the asymptotic is not predicted:

$$\frac{1}{x}\sum_{n\le x} d_k(n) \sim \frac{(\log x)^{k-1}}{(k-1)!},$$

$$\frac{1}{x}\sum_{n\le x} k^{\omega(n)} \sim C_k\frac{(\log x)^{k-1}}{(k-1)!}.$$

Behavior of $\omega(n)$ in the large deviations range $\omega(n) \approx k\log\log n$ differs from Poisson by constants.

X is a random variable with mean μ and variance σ^2.

$$\begin{aligned}
\mathbb{E}(e^{tX}) &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left(tu - \frac{(u-\mu)^2}{2\sigma^2}\right) du \\
&= e^{t\mu + t^2\sigma^2/2}\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left(-\frac{(u-\mu-t\sigma^2)^2}{2\sigma^2}\right) du \\
&= e^{t\mu + t^2\sigma^2/2}.
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Selberg: $\log|\zeta(\frac{1}{2}+it)|$ is normal with mean 0 and variance $\sim \frac{1}{2}\log\log T$.
Extrapolating Selberg suggests:

$$\begin{aligned}M_k(T) &= \int_0^T \exp(2k\log|\zeta(\tfrac{1}{2}+it)|)dt \\ &\asymp T\exp\left((2k)^2\frac{1}{4}\log\log T\right) = T(\log T)^{k^2}.\end{aligned}$$

Explains what the $2k$–th moment measures:
The $2k$–th moment of zeta is dominated by

$$\{t \in [0, T] : |\zeta(\tfrac{1}{2} + it)| \asymp (\log T)^k\},$$

and this set has measure $T/(\log T)^{k^2}$.

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k-th moment of $L(\frac{1}{2}, \chi_d)$ dominated

$$L(\tfrac{1}{2}, \chi_d) \asymp (\log |d|)^{k+1/2},$$

and the number of such $|d| \le X$ is about $X/(\log X)^{k^2/2}$.
k-th moment of $L(\frac{1}{2}, E_d)$ dominated by values of size

$$(\log X)^{k-1/2}, \quad \# \text{ such values } \asymp X/(\log X)^{k^2/2}.$$

Heuristic 3: Random matrix theory

Assume RH.

$$N(T) = \#\{0 < \gamma \le T : \zeta(\tfrac{1}{2} + i\gamma) = 0\} = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$

Let $0 < \gamma_1 \le \gamma_2 \le \dots$ denote the ordinates of zeros of $\zeta(s)$. Then

$$\gamma_n \sim \frac{2\pi n}{\log n}.$$

Write

$$\widetilde{\gamma}_n = \gamma_n \frac{\log \gamma_n}{2\pi},$$

and, on average, $\widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n$ is of size 1.

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and, on average, $\widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n$ is of size 1.

Question: What is the distribution of the spacings $\widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n$?

Given an interval (α, β) in $(0, \infty)$ what can we say about

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \le N : \widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n \in (\alpha, \beta)\}?$$

Eigenvalues of random matrices

Conjectured answer: The normalized spacings between consecutive zeros of $\zeta(s)$ behave like the normalized spacings between consecutive eigenvalues of a large random matrix.

Originates in work of Hugh Montgomery (1973) on the *Pair Correlation of zeros* of $\zeta(s)$. The connection with RMT was made during a chance encounter between Montgomery and Freeman Dyson.

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Precise version: Consider the unitary group $U(N)$, and pick a random matrix g from $U(N)$ (random with respect to Haar measure on $U(N)$). Let $e^{i\theta_1}, \ldots, e^{i\theta_N}$ denote the eigenvalues of g, arranged so that $0 \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_N < 2\pi$. Consider the distribution of the normalized angles:

$$\widetilde{\theta}_n = \frac{N}{2\pi}\theta_n.$$

These have mean spacing 1. Average their distribution over $U(N)$, and let $N \to \infty$.

Random numbers vs Random eigenvalues: Figure due to Eric Rains

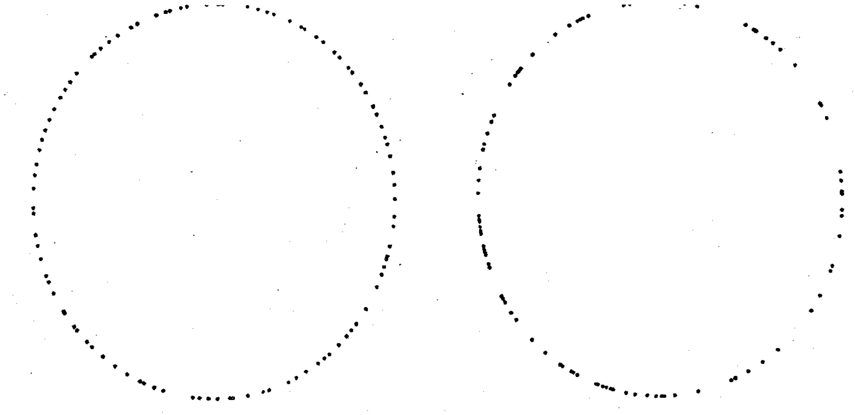

Fig. 1. a Eigenvalues of a random $U \in U(100)$. **b** 100 independent uniform points in S^1

Odlyzko's marvelous data

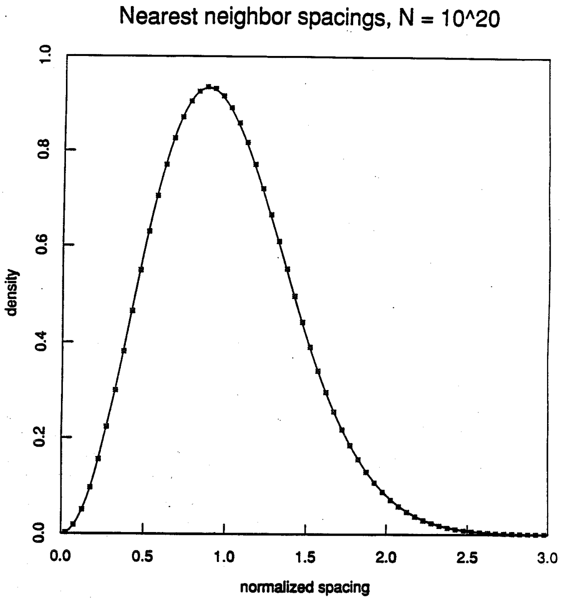

Figure 2.3.5 Probability density of the normalized spacings δ. Solid line: GUE

Odlyzko's less marvelous data!

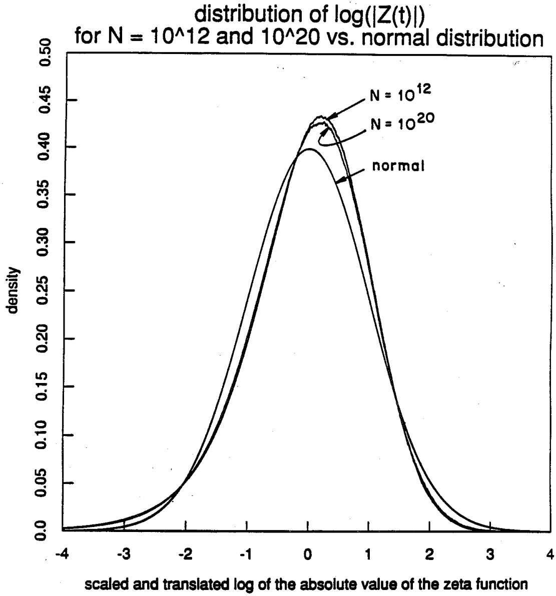

Figure 2.10.1. **Comparison of the distribution of log $|\zeta(1/2+it)|$ over two ranges of 10^6**

Keating and Snaith's insight

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How to choose N? Average spacing between consecutive zeros of $\zeta(s)$ at height T is about $2\pi/\log T$. Average spacing between two "eigen-angles" of a random matrix of size N is about $2\pi/N$. This suggests taking

$$\frac{2\pi}{\log T} \approx \frac{2\pi}{N}, \qquad \text{or} \qquad N \approx \log T.$$

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Analog of $\zeta(s)$?

Answer: Characteristic polynomial of the random matrix.

For example: one can model $\zeta(s)$ near the 10^{20}-th zero, by characteristic polynomials of random matrices of size 42.

Keating and Snaith's marvelous graph

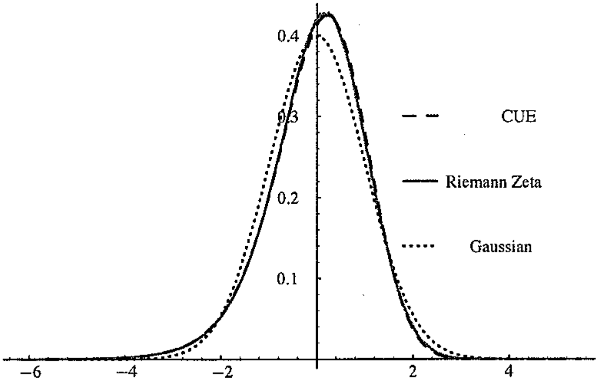

Fig. 1. The CUE value distribution for $\mathrm{Re}\log Z$ with $N = 42$, Odlyzko's data for the value distribution of $\mathrm{Re}\log\zeta(1/2+it)$ near the $10^{20\text{th}}$ zero (taken from [29]), and the standard Gaussian, all scaled to have unit variance

Back to moments

Compute the analogue in random matrix theory:

$$\int_{U(N)} |\det(I - g)|^{2k} dg = \int_{[0,1]^N} \Big| \prod_{n=1}^{N} (1 - e(\theta_n)) \Big|^{2k} \times \frac{1}{N!} \prod_{1 \le j < m \le N} |e(\theta_j) - e(\theta_m)|^2 d\theta_1 \cdots d\theta_N$$

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The Selberg integral:

$$\int_{[0,1]^n} \prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\le i<j\le n} |t_i-t_j|^{2\gamma} dt_1 \ldots dt_n$$

$$= \prod_{j=0}^{n-1} \frac{\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)\Gamma(1+(j+1)\gamma)}{\Gamma(\alpha+\beta+(n+j-1)\gamma)\Gamma(1+\gamma)}$$

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$$= \prod_{j=0}^{n-1} \frac{\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)\Gamma(1+(j+1)\gamma)}{\Gamma(\alpha+\beta+(n+j-1)\gamma)\Gamma(1+\gamma)}$$

Moment integral evaluates to

$$\prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(2k+j)}{\Gamma(k+j)^2} \sim g_k \frac{N^{k^2}}{(k^2)!}$$

Analogous conjectures in other families

Katz–Sarnak philosophy: Model low lying zeros in families of L-functions by eigen-angles nearest to 0 in random matrices chosen from appropriate classical groups.

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$L(1/2, \chi)$ for $\chi \pmod{q}$ modeled again by $U(N)$ – like for $\zeta(s)$.

$L(\frac{1}{2}, \chi_d)$ modeled by random matrices from $USp(2N)$.

Near the real axis, the density of zeros of $L(s, \chi_d)$ is $(\log |d|)/(2\pi)$ per unit length.

Eigenvalues of a matrix from $USp(2N)$: $e^{\pm i\theta_1}$, ..., $e^{\pm i\theta_N}$ arranged in ascending order $0 \leq \theta_1 \leq \ldots < \pi$.

Haar measure on $USp(2N)$:

$$\text{Normalizing constant} \times \prod_{1 \leq i < j \leq N} \left(\cos\theta_i - \cos\theta_j\right)^2 \prod_{k=1}^{N} (\sin\theta_k)^2 d\theta_k.$$

Conjecture: Behavior of $\gamma_1 \frac{\log |d|}{2\pi}$ is exactly the same as the behavior of $\theta_1 \frac{2N}{2\pi}$.

Note: in symplectic family – zeros near 1/2 are "repelled."

Keating–Snaith Scaling: d of size X corresponds to N of size $\log\sqrt{X}$.

Conjecture:

$$\frac{1}{\#\{d\}}\sum_{|d|\le X}^{\flat} L(\tfrac{1}{2},\chi_d)^k \sim f_k a_k (\log\sqrt{X})^{k(k+1)/2}$$

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k-th moment of characteristic polynomial averaged over $USp(2N)$ with respect to the Haar measure.

Selberg integral again!

$$2^{2Nk}\prod_{j=1}^{N}\frac{\Gamma(1+N+j)\Gamma(1/2+k+j)}{\Gamma(1/2+j)\Gamma(1+k+N+j)} \sim f_k \frac{N^{k(k+1)/2}}{(k(k+1)/2)!}$$

$$f_k = \frac{(k(k+1)/2)!}{\prod_{j=1}^{k}(2j-1)!!}.$$

Heuristic 4: Symmetrization of diagonal terms

Even primitive Dirichlet characters $\chi \pmod{q}$.
Introduce "shifts" $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k$: want to understand

$$\sideset{}{^*}\sum_{\substack{\chi \pmod{q} \\ \chi(-1)=1}} \prod_{j=1}^{k} \Lambda(\tfrac{1}{2}+\alpha_j,\chi)\Lambda(\tfrac{1}{2}-\beta_j,\overline{\chi})$$

$\Lambda(\frac{1}{2}+s,\chi) = (q/\pi)^{\frac{s}{2}}\Gamma(1/4+s/2)L(\frac{1}{2}+s,\chi) = \epsilon_\chi \Lambda(\frac{1}{2}-s,\overline{\chi}),\ \ \epsilon_\chi\epsilon_{\overline{\chi}} = 1$

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$2k$–th moment corresponds to setting all $\alpha_j = \beta_j = 0$.
Observe: Conjecture symmetric in $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)$.
Proof: Obviously symmetric in $(\alpha_1, \ldots, \alpha_k)$ and $(\beta_1, \ldots, \beta_k)$.

$$\begin{aligned}\Lambda(\tfrac{1}{2}+\alpha_i,\chi)\Lambda(\tfrac{1}{2}-\beta_j,\overline{\chi}) &= \epsilon_\chi \Lambda(\tfrac{1}{2}-\alpha_i,\overline{\chi})\epsilon_{\overline{\chi}}\Lambda(\tfrac{1}{2}+\beta_j,\chi) \\ &= \Lambda(\tfrac{1}{2}+\beta_j,\chi)\Lambda(\tfrac{1}{2}-\alpha_i,\overline{\chi})\end{aligned}$$

$$\delta(\alpha,\beta) = \frac{1}{2}\sum_{j=1}^{k}(\alpha_j - \beta_j), \quad G(\alpha,\beta) = \prod_{j=1}^{k}\Gamma(\tfrac{1}{2} + \tfrac{\alpha_j}{2})\Gamma(\tfrac{1}{2} - \tfrac{\beta_j}{2})$$

$$\sigma(n;\alpha) = \sum_{n=n_1\cdots n_k} n_1^{-\alpha_1}\cdots n_k^{-\alpha_k}$$

$$\prod_{j=1}^{k} L(s+\alpha_j,\chi) = \sum_{n=1}^{\infty}\frac{\sigma(n;\alpha)}{n^s}\chi(n)$$

$$\prod_{j=1}^{k} L(s-\beta_j,\overline{\chi}) = \sum_{m=1}^{\infty}\frac{\sigma(m;-\beta)}{m^s}\overline{\chi}(m).$$

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Diagonal Contribution $=(q/\pi)^{\delta(\alpha,\beta)}G(\alpha,\beta)$ times

$$\sum_{(n,q)=1}\frac{\sigma(n;\alpha)\sigma(n;-\beta)}{n^{2s}}$$

$$=\prod_{p\nmid q}\left(1+\frac{1}{p^{2s}}(p^{-\alpha_1}+\ldots+p^{-\alpha_k})(p^{\beta_1}+\ldots+p^{\beta_k})+\ldots\right)$$

$$\prod_{p\nmid q}\left(1+\frac{1}{p^{2s}}(p^{-\alpha_1}+\ldots+p^{-\alpha_k})(p^{\beta_1}+\ldots+p^{\beta_k})+\ldots\right)$$

$$=\mathcal{A}(s;\alpha,\beta)\mathcal{Z}(s;\alpha,\beta),$$

where

$$\mathcal{Z}(s;\alpha,\beta)=\prod_{j,\ell=1}^{k}\zeta(2s+\alpha_j-\beta_\ell).$$

Diagonal contribution:

$$\left(\frac{q}{\pi}\right)^{\delta(\alpha,\beta)}G(\alpha,\beta)\mathcal{A}(\tfrac{1}{2};\alpha,\beta)\mathcal{Z}(\tfrac{1}{2};\alpha,\beta).$$

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Symmetry? Can permute $(\alpha_1,\ldots,\alpha_k)$ and $(\beta_1,\ldots,\beta_k)$ but not allowed to switch α's and β's.

Conjecture: symmetrize this answer

$$\sum_{\pi\in S_{2k}/(S_k\times S_k)}\left(\frac{q}{\pi}\right)^{\delta(\pi(\alpha,\beta))}G(\pi(\alpha,\beta))\mathcal{A}(\tfrac{1}{2};\pi\alpha,\pi\beta)\mathcal{Z}(\tfrac{1}{2};\pi\alpha,\pi\beta).$$

For example, $k = 3$ is a sum of 20 terms:

- One "no swap" term $(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3)$.
- Nine "one swap" terms e.g. $(\beta_1, \alpha_2, \alpha_3; \alpha_1, \beta_2, \beta_3)$.
- Nine "two swap" terms e.g. $(\beta_1, \beta_2, \alpha_3; \alpha_1, \alpha_2, \beta_3)$.
- One "all swap" term: $(\beta_1, \beta_2, \beta_3; \alpha_1, \alpha_2, \alpha_3)$.

Each individual term has singularities. E.g. "no swap" term has singularities when $\alpha_i = \beta_j$.

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The sum is regular!

Now let all the shifts $\to 0$.

Similar "recipe" for moments in families of L-functions: Conrey, Farmer, Keating, Rubinstein, & Snaith.

Works only for integral moments: $k \in \mathbb{N}$.

Identifies all lower order terms in conjecture for moments.

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} dt \approx \int_0^T P_k\Big(\frac{\log t}{2\pi}\Big) dt$$

for a polynomial P_k of degree k^2.

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k} dt \approx \int_0^T P_k\Big(\frac{\log t}{2\pi}\Big) dt$$

$$P_k(x) = \frac{(-1)^k}{k!^2} \frac{1}{(2\pi i)^{2k}} \int_{z_1,\dots,z_{2k}} \mathcal{A}(z_1,\dots,z_{2k}) \prod_{i,j=1}^{k} \zeta(1+z_i-z_{k+j})$$
$$\times \Delta(z_1,\dots,z_{2k})^2 \exp\Big(\frac{x}{2}\sum_{j=1}^{k}(z_j-z_{k+j})\Big) \prod_{j=1}^{2k} \frac{dz_j}{z_j^{2k}}$$

where the integrals are over small circles centered at 0 and

$$\Delta(z_1,\dots,z_{2k}) = \prod_{1\le i\le j\le 2k} (z_j - z_i).$$

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where the integrals are over small circles centered at 0 and

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$$P_3(x) \approx 5.7\times 10^{-6}x^9 + 4\times 10^{-4}x^8 + 1.1\times 10^{-2}x^7$$
$$+ 0.14x^6 + x^5 + 3.98x^4 + 8.6x^3 + 10.2x^2 + 6.59x + 0.91$$

Analogous symmetrization in other families

E.g. consider – restricting to positive discriminants –

$$\sum_{d \le X}^{\flat} \Lambda(\tfrac{1}{2} + \alpha_1, \chi_d) \cdots \Lambda(\tfrac{1}{2} + \alpha_k, \chi_d),$$

$$\Lambda(\tfrac{1}{2} + s, \chi_d) = (d/\pi)^{s/2} \Gamma(1/4 + s/2) L(\tfrac{1}{2} + s, \chi_d) = \Lambda(\tfrac{1}{2} - s, \chi_d).$$

Answer must be symmetric under all $\alpha_j \to \epsilon_j \alpha_j$ where $\epsilon_j = \pm 1$.

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Answer must be symmetric under all $\alpha_j \to \epsilon_j\alpha_j$ where $\epsilon_j = \pm 1$.

Diagonal contribution:

$$\left(\frac{d}{\pi}\right)^{\sum \alpha_j/2} \prod_j \Gamma(\tfrac{1}{4}+\tfrac{\alpha_j}{2}) \sum_{\substack{n_1,\ldots,n_k \\ n_1\cdots n_k=\Box \\ (n_j,d)=1}} \prod_j \frac{1}{n_j^{\frac{1}{2}+\alpha_j}}.$$

The sum over n_1, …, n_k can be written as

$$\mathcal{A}(\alpha_1\ldots,\alpha_k) \prod_{1\leq i\leq j\leq k} \zeta(1+\alpha_i+\alpha_j).$$

Symmetrize this expression by summing over all 2^k choices of ϵ_j.

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Example: $\zeta(\frac{1}{2}+it)$ — think of the integral as having size T.
Approximate functional equation:

$$\zeta(s)^k \approx \sum_{n \le T^{k/2}} \frac{d_k(n)}{n^s} + \left(\pi^{s-1/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\right)^k \sum_{n \le T^{k/2}} \frac{d_k(n)}{n^{1-s}}$$

Allows evaluation of second and fourth moments.

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Allows evaluation of second and fourth moments.

Important/interesting to go beyond this rule of thumb.

Possiblities: (i) evaluate a higher moment; (ii) make the family smaller (e.g. integrate over short interval); (iii) get a power saving in the error term; (iv) compute moment with a short Dirichlet polynomial thrown in (an "amplifier").

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2 dt = \mathrm{MT}+O(T^{\frac{1}{3}+\epsilon}); \quad \int_T^{T+T^{\frac{2}{3}}} |\zeta(\tfrac{1}{2}+it)|^4 dt \ll T^{\frac{2}{3}+\epsilon}$$

Analogues in many different families. Can be hard to make the rule of thumb work.

$$\sum_{\chi \pmod{q}}^{*} |L(\tfrac{1}{2}, \chi)|^4 = \mathrm{MT} + O(q^{1-\delta})$$ Matt Young

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Asymptotic large sieve: Conrey, Iwaniec & S., Chandee, Li, Matomaki & Radziwill:

$$\sum_{q\le Q} \sum_{\chi \pmod q}^{*} |L(\tfrac{1}{2},\chi)|^6, \quad \sum_{q\le Q} \sum_{\chi \pmod q}^{*} \int_{-1}^{1} |L(\tfrac{1}{2}+it,\chi)|^8.$$

$$\sum_{\chi \pmod q}^{*} L(\tfrac{1}{2}, f\times\chi)\overline{L(\tfrac{1}{2}, g\times\chi)} = \mathrm{MT} + O(q^{1-\delta}) \text{ Kowalski-Michel-Sawin}$$

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$$\sum_{|d|\le X}^{\flat} L(\tfrac{1}{2},\chi_d)^4, \quad \sum_{|d|\le X}^{\flat} L(\tfrac{1}{2},E_d)^2$$ (S., S. & Young, Florea)

Two Principles

Lower bounds Principle: If one can compute the first moment (plus epsilon) then can get the right lower bound for all larger moments. (Rudnick & S., Radziwill & S.)
"Plus epsilon" means one should be able to compute the moment multiplied by a short Dirichlet polynomial.

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Upper bounds on GRH: In general families one can establish the conjectured upper bound. (S., plus sharp refinement by Adam Harper)

Outstanding open problems.

1. Find correct lower bounds for small moments (e.g. less than the first).
Uniform such bounds as $k \to 0+$ imply positive proportion of non vanishing.
Known in some cases due to Chandee & Li; not known for the family of quadratic twists of E.

2. Establish upper bounds for large moments – related to sub-convexity, Lindelöf hypothesis.

Upper bounds Principle

Theorem: (Heap, Radziwill & S.) For $0 \leq k \leq 2$

$$\int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^{2k} dt \ll T(\log T)^{k^2}.$$

Previous work:

- $k = 1/n$ by Heath-Brown (Ramachandra $k < 2$ assuming RH)
- $k = 1 + 1/n$ by Bettin, Chandee & Radziwill

Key input: (Deshouillers & Iwaniec, Hughes & Young, Bettin, Bui, Li, & Radziwill)

$$\int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^4 |A(t)|^2 dt, \quad A(t) = \sum_{n \leq N} a(n) n^{-1/2+it}.$$

Quadratic twists of an elliptic curve

E – elliptic curve over $\mathbb{Q}$ with conductor N.

$\mathcal{E}$ – fundamental discriminants d with root number of E_d being 1.

Theorem: Radzwill & S. For $0 \le k \le 1$

$$\sideset{}{^\flat}\sum_{\substack{|d|\le X \\ d\in\mathcal{E}}} L(\tfrac{1}{2}, E_d)^k \ll X(\log X)^{k(k-1)/2},$$

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Work of Young; Kowalski, Michel, & Sawin should allow similar results for

$$\sum_{\chi \pmod q}^{*} |L(\tfrac{1}{2}, \chi)|^{2k}, \quad 0 \le k \le 2$$

$$\sum_{\chi \pmod q}^{*} |L(\tfrac{1}{2}, f \times \chi)|^{2k}, \quad 0 \le k \le 1.$$

$$\sum_{|d|\le X}^{\flat} |L(\tfrac{1}{2}, \chi_d)|^k \ll X(\log X)^{k(k+1)/2}, \quad 0 \le k \le 2.$$ Larger range??

Proof for quadratic twists of an elliptic curve

Iterative scheme inspired by the "pure Brun sieve."

Closely related to Harper's work on sharp conditional bounds for moments.

$$\ell_1 = 2\lceil 100 \log\log X \rceil, \qquad \ell_{j+1} = 2\lceil 100 \log \ell_j \rceil,$$

stopping at the largest R with $\ell_R > 10^4$.

$$\mathcal{P}_1(d) = \sum_{p \le X^{1/\ell_1^2}} \frac{a(p)}{\sqrt{p}} \chi_d(p),$$

$$\mathcal{P}_j(d) = \sum_{X^{1/\ell_{j-1}^2} \le p \le X^{1/\ell_j^2}} \frac{a(p)}{\sqrt{p}} \chi_d(p).$$

Ideas:

Think of $\exp(\mathcal{P}_1(d) + \ldots + \mathcal{P}_R(d))$ as being like $L(\frac{1}{2}, E_d)(\log |d|)^{\frac{1}{2}}$.

Work with Taylor series approximations to $\exp(\mathcal{P}_j(d))$ – fewer terms needed as j gets larger.

A general inequality

Recall: $E_\ell(x) = \sum_{j=0}^{\ell} x^j/j!$. If ℓ is even and $x \le \ell/e^2$ then

$$e^x \le \left(1 + \frac{e^{-\ell}}{16}\right) E_\ell(x).$$

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Lemma: $y \ge 0$. $x_1, \ldots, x_R$ real numbers. $\ell_1, \ldots, \ell_R$ positive even. Then, for any $0 \le k \le 1$

$$y^k \le Cky \prod_{j=1}^{R} E_{\ell_j}((k-1)x_j) + C(1-k) \prod_{j=1}^{R} E_{\ell_j}(kx_j)$$

$$+ \sum_{r=0}^{R-1} \Big(Cky \prod_{j=1}^{r} E_{\ell_j}((k-1)x_k) + C(1-k) \prod_{j=1}^{r} E_{\ell_j}(kx_j) \Big) \left(\frac{e^2 x_{r+1}}{\ell_{r+1}} \right)^{\ell_{r+1}},$$

where $C = \exp((e^{-\ell_1} + \ldots + e^{-\ell_R})/16)$.

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Plan: Apply this with

$$y = L(\tfrac{1}{2}, E_d)(\log |d|)^{\frac{1}{2}}; \quad x_j = \mathcal{P}_j(d).$$

Proof of Lemma

W.H. Young's inequality:

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Suppose $x_j \le \ell_j/e^2$ for all $1 \le j \le R$.

$y^k \le ky \exp((k-1)(x_1 + \ldots + x_R)) + (1-k)\exp(k(x_1 + \ldots + x_R))$

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$\exp(kx_j) \le \left(1+\frac{e^{-\ell_j}}{16}\right) E_{\ell_j}(kx_j), \;\; \exp((k-1)x_j) \le \left(1+\frac{e^{-\ell_j}}{16}\right) E_{\ell_j}((k-1)x_j).$

Therefore, with $C = \exp((e^{-\ell_1} + \ldots + e^{-\ell_R})/16)$

$$y^k \le Cky \prod_{j=1}^{R} E_{\ell_j}((k-1)x_j) + C(1-k) \prod_{j=1}^{R} E_{\ell_j}(kx_j).$$

Gives first term in bound of Lemma.

Suppose $0 \leq r \leq R-1$ such that $x_j \leq \ell_j/e^2$ for $j \leq r$, but $x_{r+1} > \ell_{r+1}/e^2$.

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Young's inequality gives

$$\begin{aligned} y^k &\le ky \exp((k-1)(x_1 + \ldots + x_r)) + (1-k) \exp(k(x_1 + \ldots + x_r)) \\ &\le Cky \prod_{j=1}^{r} E_{\ell_j}((k-1)x_j) + C(1-k) \prod_{j=1}^{r} E_{\ell_j}(kx_j) \\ &\le \left(Cky \prod_{j=1}^{r} E_{\ell_j}((k-1)x_j) + C(1-k) \prod_{j=1}^{r} E_{\ell_j}(kx_j) \right) \left(\frac{e^2 x_{r+1}}{\ell_{r+1}} \right)^{\ell_{r+1}}. \end{aligned}$$

Last inequality holds because $(e^2 x_{r+1}/\ell_{r+1})^{\ell_{r+1}}$ is always positive, and is ≥ 1 in the case $x_{r+1} \ge \ell_{r+1}/e^2$.

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Lemma follows upon summing over all these possibilities.

$$y = L(\tfrac{1}{2}, E_d)(\log|d|)^{\frac{1}{2}}; \quad x_j = \mathcal{P}_j(d).$$

$$\mathcal{A}_j(d) = E_{\ell_j}((k-1)\mathcal{P}_j(d)), \quad \mathcal{B}_j(d) = E_{\ell_j}(k\mathcal{P}_j(d))$$

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Lemma bounds $L(\frac{1}{2}, E_d)^k(\log|d|)^{\frac{k}{2}}$ by

$$\ll L(\tfrac{1}{2}, E_d)(\log|d|)^{\frac{1}{2}}\Big(\prod_{j=1}^{R}\mathcal{A}_j(d) + \sum_{r=0}^{R-1}\prod_{j=1}^{r}\mathcal{A}_j(d)\Big(\frac{e^2\mathcal{P}_{r+1}(d)}{\ell_{r+1}}\Big)^{\ell_{r+1}}\Big)$$

$$+\Big(\prod_{j=1}^{R}\mathcal{B}_j(d) + \sum_{r=0}^{R-1}\prod_{j=1}^{r}\mathcal{B}_j(d)\Big(\frac{e^2\mathcal{P}_{r+1}(d)}{\ell_{r+1}}\Big)^{\ell_{r+1}}\Big)$$

Note $\mathcal{A}_j$ and $\mathcal{B}_j$ are Dirichlet polynomials of length $\le X^{1/\ell_j}$. So

$$\prod_{j=1}^{R}\mathcal{A}_j(d), \ \prod_{j=1}^{R}\mathcal{B}_j(d), \ \prod_{j=1}^{r}\mathcal{A}_j(d)\mathcal{P}_{r+1}(d)^{\ell_{r+1}}, \ \prod_{j=1}^{r}\mathcal{B}_j(d)\mathcal{P}_{r+1}(d)^{\ell_{r+1}}$$

are all short Dirichlet polynomials of length $\le X^{1/1000}$.

Proposition:

$$\sum_{\substack{|d|\le X\\ d\in\mathcal{E}}}^{\flat}\Big(\prod_{j=1}^{R}\mathcal{B}_j(d)+\sum_{r=0}^{R-1}\prod_{j=1}^{r}\mathcal{B}_j(d)\Big(\frac{e^2\mathcal{P}_{r+1}(d)}{\ell_{r+1}}\Big)^{\ell_{r+1}}\Big)\ll X(\log X)^{\frac{k^2}{2}}$$

$$\sum_{\substack{|d|\le X\\ d\in\mathcal{E}}}^{\flat}L(\tfrac{1}{2},E_d)(\log|d|)^{\frac{1}{2}}\Big(\prod_{j=1}^{R}\mathcal{A}_j(d)+\sum_{r=0}^{R-1}\prod_{j=1}^{r}\mathcal{A}_j(d)\Big(\frac{e^2\mathcal{P}_{r+1}(d)}{\ell_{r+1}}\Big)^{\ell_{r+1}}\Big)$$

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- Compute averages of short Dirichlet polynomials.
- Compute first moment times short Dirichlet polynomial.

Idea behind Proposition

Can focus on diagonal contributions — here, square terms.
Primes in $\mathcal{P}_j(d)$ are disjoint for different j — behave independently on average.

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$$\mathbb{E}_d(\mathcal{B}_j(d)) \approx \prod_{X^{1/\ell_{j-1}^2} \le p \le X^{1/\ell_j^2}} \left(1 + \frac{k^2}{2}\frac{a(p)^2}{p} + \dots\right)$$

$$\mathbb{E}_d\left(\mathcal{P}_{r+1}(d)^{\ell_{r+1}}\right) \approx \frac{\ell_{r+1}!}{2^{\ell_{r+1}/2}(\ell_{r+1}/2)!}\left(\sum_{X^{1/\ell_r^2} \le X^{1/\ell_{r+1}^2}} \frac{a(p)^2}{p}\right)^{\ell_{r+1}/2}$$

$$\le \ell_{r+1}^{\ell_{r+1}/2}(2\log \ell_r)^{\ell_{r+1}/2} \le \left(\frac{\ell_{r+1}}{10}\right)^{\ell_{r+1}}$$

Recall: $\ell_{j+1} = 2\lfloor 100 \log \ell_j \rfloor$.

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$$\sideset{}{^\flat}\sum_{\substack{|d| \le X \\ d \in \mathcal{E}}} \prod_{j=1}^{r} \mathcal{B}_j(d) \Big(\Big(\frac{e^2 \mathcal{P}_{r+1}(d)}{\ell_{r+1}}\Big)^{\ell_{r+1}}\Big) \ll \left(\log X^{\frac{1}{\ell_r^2}}\right)^{\frac{k^2}{2}} \left(\frac{e}{10}\right)^{\ell_{r+1}}.$$

Similar calculation for terms involving $L(\frac{1}{2}, E_d)$.

$$\mathbb{E}_d(L(\tfrac{1}{2}, E_d)\mathcal{A}_j(d))$$
$$\approx \prod_{X^{1/\ell_{j-1}^2} \le p \le X^{1/\ell_j^2}} \left(1 + \frac{(k-1)a(p)}{\sqrt{p}} \frac{a(p)}{\sqrt{p}} + \frac{(k-1)^2}{2} \frac{a(p)^2}{p} + \dots\right)$$

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Conclude

$$\sum_{\substack{|d|\le X \\ d\in\mathcal{E}}}^{\flat} L(\tfrac{1}{2}, E_d)\prod_{j=1}^{r}\mathcal{A}_j(d)\Big(\frac{e^2\mathcal{P}_{r+1}(d)}{\ell_{r+1}}\Big)^{\ell_{r+1}}\Big) \ll \Big(\log X^{\frac{1}{\ell_r^2}}\Big)^{\frac{k^2-1}{2}}\Big(\frac{e}{10}\Big)^{\ell_{r+1}}.$$

Completes sketch of Proposition.

Lower bounds Principle

Story for $\zeta(s)$

Titchmarsh: $k \in \mathbb{N}$

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^{2k} e^{-t/T} dt \gg_k T(\log T)^{k^2}.$$

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Conrey & Ghosh: Elegant proof for $k \in \mathbb{N}$

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k} dt \geq T \sum_{n \leq T} \frac{d_k(n)^2}{n} \sim (1+o(1)) a_k T(\log T)^{k^2}.$$

Idea: n^{it} for $n \leq T^{1-\epsilon}$ is a family of nearly orthogonal vectors:

$$\frac{1}{T}\int_0^T n^{it} m^{-it} dt \approx \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}$$

Can also compute their inner product with $\zeta(s)^k$: for $n \leq T^{1-\epsilon}$

$$\frac{1}{T}\int_0^T \zeta(\tfrac{1}{2}+it)^k n^{it} dt \approx \frac{1}{T}\int_0^T \sum_{m \leq T^k} \frac{d_k(m)}{m^{\frac{1}{2}+it}} n^{it} dt \approx \frac{d_k(n)}{\sqrt{n}}.$$

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Now invoke Bessel's inequality.
Equivalently: Use for any $A(s) = \sum_{n \le T^{1-\epsilon}} a(n) n^{-s}$

$$\Big| \int_0^T \zeta(\tfrac{1}{2}+it)^k A(\tfrac{1}{2}-it) dt \Big|^2 \le \Big(\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k} dt \Big) \Big(\int_0^T |A(\tfrac{1}{2}+it)|^2 dt \Big).$$

Optimal choice: $a(n) = d_k(n)$.

Analogues for families?

Methods for ζ don't extend automatically.

How to compute, for small n,

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How to handle fractional power $B(d)^{\frac{k}{k-1}}$?

Trick: Just choose $B(d) = A(d)^{k-1}$ for a Dirichlet polynomial A.

$x = X^{1/(2k)}$, and put $A(d) = \sum_{n \le x} \chi_d(n)/\sqrt{n}$.
Evaluate

$$\sideset{}{^\flat}\sum_{|d| \le X} A(d)^k, \quad \sideset{}{^\flat}\sum_{|d| \le X} L(\tfrac{1}{2}, \chi_d) A(d)^{k-1},$$

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and use Hölder.

$$A(d)^k = \sum_{n \le x^k} \frac{d_k(n;x)}{\sqrt{n}} \chi_d(n); \qquad d_k(n) = \sum_{\substack{m_1 \cdots m_k = n \\ m_i \le x}} 1.$$

Only diagonal terms matter:

$$\sideset{}{^\flat}\sum_{|d| \le X} A(d)^k = \sum_{n \le x^k} \frac{d_k(n;x)}{\sqrt{n}} \sideset{}{^\flat}\sum_{|d| \le X} \left(\frac{d}{n}\right) \ll X \sum_{\substack{n \le x^k \\ n = m^2}} \frac{d_k(m^2)}{m}.$$

Conclude

$$\sideset{}{^\flat}\sum_{|d| \le X} A(d)^k \ll X (\log X)^{k(k+1)/2}.$$

Want a similar lower bound for

$$\sum_{|d|\le X}^{\flat} L(\frac{1}{2},\chi_d)A(d)^{k-1} = \sum_{n\le x^{k-1}} \frac{d_{k-1}(n;x)}{\sqrt{n}} \sum_{|d|\le X}^{\flat} L(\tfrac{1}{2},\chi_d)\chi_d(n)$$

$$\gg X \sum_{n\le x^{k-1}} \frac{d_{k-1}(n;x)}{\sqrt{n}} \sum_{\substack{m\le X \\ nm=\square}} \frac{1}{\sqrt{m}}$$

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Restrict to $n \le x$ so that $d_{k-1}(n;x) = d_{k-1}(n)$.
Write $n = n_1 n_2^2$ with n_1 squarefree.

$$\sum_{\substack{m\le X\\ nm=\square}} \frac{1}{\sqrt{m}} = \frac{1}{\sqrt{n_1}} \sum_{r\le\sqrt{X/n_1}} \frac{1}{r} \gg \frac{\log X}{\sqrt{n_1}}$$

So get lower bound $\gg X(\log X)^{k(k+1)/2}$:

$$\gg X\log X \sum_{n\le x} \frac{d_{k-1}(n)}{\sqrt{n}\sqrt{n_1}} \gg X\log X \prod_{p\le x}\Big(1+\frac{k-1}{\sqrt{p}\sqrt{p}}+\frac{(k-1)k/2}{p}\Big).$$

Theorem: (Rudnick & S) For all $k \in \mathbb{N}$

$$\sum_{|d|\le X}^{\flat} L(\tfrac{1}{2},\chi_d)^k \gg \Big(\sum_{|d|\le X}^{\flat} L(\tfrac{1}{2},\chi_d)A(d)^{k-1}\Big)^k \Big/ \Big(\sum_{|d|\le X}^{\flat} A(d)^k\Big)^{k-1}$$
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Method extends to give correct lower bounds for all rational $k \ge 1$.
If $k = r/s$ start with

$$\sum_{|d|\le X}^{\flat} L(\tfrac{1}{2},\chi_d)(B(d)^s)^{k-1},$$

with $(\zeta(w)^z = \sum_n d_z(n)/n^w)$

$$B(d) = \sum_{n\le x} \frac{d_{1/s}(n)}{\sqrt{n}} \chi_d(n).$$

Take $x = X^{1/(2r)}$. Then $B(d)^{s(k-1)} = B(d)^{r-s}$ is still a short Dirichlet polynomial, and can evaluate this. Get

$$\sum_{|d|\le X}^{\flat} L(\tfrac{1}{2}, \chi_d)(B(d)^s)^{k-1} \gg_k X(\log x)^{k(k+1)/2}.$$

Similarly, $B(d)^{sk} = B(d)^r$ is a short Dirichlet polynomial, and so can evaluate

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By Hölder:

$$\sum_{|d|\le X}^{\flat} L(\tfrac{1}{2}, \chi_d)^k \gg \frac{(X(\log x)^{k(k+1)/2})^k}{(X(\log x)^{k(k+1)/2})^{k-1}} \gg_{r,s} X(\log X)^{k(k+1)/2}.$$

Note: Implied constant depends not just on k, but on height of $k = r/s$. "Discontinuous" in k.

Radziwill & S: Refinement which provides “continuous” lower bounds in k.
Obtain lower bounds of the right order for all $k \geq 1$.

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Alternative treatment of lower bounds — in progress, Heap, Radziwill & S.

$$\Big(\sum_{|d| \leq X}^{\flat} L(\tfrac{1}{2}, \chi_d) \exp(\mathcal{P}(d)(k-1)) \Big)^k \leq \Big(\sum_{|d| \leq X}^{\flat} L(\tfrac{1}{2}, \chi_d)^k \Big) \Big(\sum_{|d| \leq X}^{\flat} \exp(k \mathcal{P}(d)) \Big)^{(k-1)/k}.$$

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Theorem: For all $0 \leq k \leq 2$

$$\int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^{2k} dt \asymp T(\log T)^{k^2}.$$