Conditional upper bounds for moments & Extreme values

K. Soundararajan

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The Riemann hypothesis and moments

Riemann hypothesis $\implies$ Lindelöf Hypothesis.

Quantitative form: Littlewood, Chandee & S.

$$|\zeta(\tfrac{1}{2}+it)| \le \exp\left(\left(\frac{\log 2}{2}+o(1)\right)\frac{\log t}{\log\log t}\right).$$

Related: Littlewood, Carneiro, Chandee & Milinovich

$$|S(t)| \le \left(\frac{1}{4}+o(1)\right)\frac{\log t}{\log\log t}.$$

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What does RH imply about the frequency of large values of $|\zeta(\frac{1}{2}+it)|$?

Does RH give better upper bounds on moments?

Analogues in families of L–functions?

Classical: RH gives moments to the right of the critical line:

$$\int_0^T |\zeta(\sigma+it)|^{2k} dt \sim T\sum_{n=1}^{\infty}\frac{d_k(n)^2}{n^{2\sigma}}.$$

$$\mathcal{S}(T,V) = \{t \in [T,2T] : \log|\zeta(\tfrac{1}{2}+it)| \geq V\}$$

Selberg's Theorem: meas $\mathcal{S}(T,V)$ when V is of size $\sqrt{\log\log T}$.
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Theorem: Assume RH. In the range
$\sqrt{\log\log T}\leq V\leq o(\log_2 T\log_3 T)$

$$\text{meas } \mathcal{S}(T,V)\ll T\exp\Big(-(1+o(1))\frac{V^2}{\log\log T}\Big).$$

For larger V, for some $c>0$

$$\text{meas } \mathcal{S}(T,V)\ll T\exp(-cV\log V).$$

Corollary: On RH, $M_k(T)\ll T(\log T)^{k^2+\epsilon}$ for any $\epsilon>0$.

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Theorem (Harper): On RH

$$\int_0^T|\zeta(\tfrac{1}{2}+it)|^{2k}dt\ll T(\log T)^{k^2}.$$

On GRH:

$$\sum_{\chi \pmod q} |L(\tfrac{1}{2},\chi)|^{2k} \ll q(\log q)^{k^2}$$

$$\sum_{|d|\le X} L(\tfrac{1}{2},\chi_d)^k \ll X(\log X)^{k(k+1)/2}$$

$$\sum_{|d|\le X} L(\tfrac{1}{2},f\times\chi_d)^k \ll X(\log X)^{k(k-1)/2}$$

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Milinovich/Kirila:

$$\sum_{|\rho|\le T} |\zeta'(\rho)|^{2k} \ll T(\log T)^{k^2+2k+1}.$$

Najnudel:

$$\int_T^{2T} \exp(2k\pi S(t))dt \ll T(\log T)^{k^2+\epsilon}.$$

The key idea

Selberg in his work on CLT — approximations for $\log \zeta(\frac{1}{2}+it)$ in terms of $\sum_{p\le x} 1/p^{1/2+it}$.
For $\operatorname{Im}\ \log \zeta(\frac{1}{2}+it)$ this can be done. More complicated for $\log|\zeta(\frac{1}{2}+it)|$ — singularities coming from zeros of $\zeta(s)$.

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For any $t \in \mathbb{R}$, the function $|\xi(\sigma + it)|$ is increasing in $\sigma \ge 1/2$.

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Proof: If RH is true then

$$|\xi(\sigma+it)| = \prod_\rho \Big|1-s/\rho\Big| = \prod_\rho \frac{|(\sigma-1/2)+i(t-\gamma)|}{|\rho|}$$

and each term in the product is increasing in $\sigma\ge 1/2$.

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and each term in the product is increasing in $\sigma \ge 1/2$.

Unconditional version: $|\xi(\sigma + it)|$ is increasing in $\sigma \ge 1$ — useful fact!

Quantitative version of the key idea

Proposition: Assume RH. T large, $2 \le x \le T^2$, $t \in [T, 2T]$. Then with $\sigma_0 = 1/2 + 1/\log x$,

$$\log|\zeta(\tfrac{1}{2} + it)| \le \mathrm{Re} \sum_{n \le x} \frac{\Lambda(n)}{n^{\sigma_0 + it} \log n} \frac{\log x/n}{\log x} + \frac{\log T}{\log x}.$$

In rough form:

$$\log|\zeta(\tfrac{1}{2} + it)| \le \mathrm{Re} \sum_{p \le x} \frac{1}{p^{\frac{1}{2} + it}} + \frac{\log T}{\log x}.$$

Key feature: flexibility in choosing x. For example, choosing $x = \log T$ gives Littlewood's quantitative RH $\implies$ LH.

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Analogues for L-functions in families: for example

$$\log L(\tfrac{1}{2}, \chi_d) \le \sum_{p \le x} \frac{\chi_d(p)}{\sqrt{p}} + \frac{1}{2} \sum_{p \le \sqrt{x}} \frac{\chi_d(p^2)}{p} + \frac{\log |d|}{\log x}$$

Proof of the Proposition

By RH, $|\xi(\frac{1}{2}+it)| \leq |\xi(\sigma_0+it)|$. Using Stirling:

$$\log|\zeta(\frac{1}{2}+it)| \leq \log|\zeta(\sigma_0+it)| + \frac{\log T}{2\log x}.$$

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Lemma (a la Selberg): Up to negligible terms

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n\le x} \frac{\Lambda(n)}{n^s}\frac{\log(x/n)}{\log x} + \frac{1}{\log x}\left(\frac{\zeta'}{\zeta}(s)\right)' + \frac{1}{\log x}\sum_{\rho}\frac{x^{\rho-s}}{(\rho-s)^2}.$$

Proof: Start with

$$\frac{1}{2\pi i}\int_{(c)} -\frac{\zeta'}{\zeta}(s+w)\frac{x^w}{w^2}dw = \sum_{n\le x}\frac{\Lambda(n)}{n^s}\log(x/n).$$

Move line to the left and compute residues:

$$-\frac{\zeta'}{\zeta}(s)\log x - \left(\frac{\zeta'}{\zeta}(s)\right)' - \sum_{\rho}\frac{x^{\rho-s}}{(\rho-s)^2}$$

Use the Lemma with $s = \sigma + it$ and integrate σ from σ_0 to infinity. With $s_0 = \sigma_0 + it$:

$$\log|\zeta(s_0)| \approx \mathrm{Re}\Big(\sum_{n \le x} \frac{\Lambda(n)}{n^{s_0}\log n} \frac{\log(x/n)}{\log x} - \frac{1}{\log x} \frac{\zeta'}{\zeta}(s_0) + \frac{1}{\log x} \sum_{\rho} \int_{\sigma_0}^{\infty} \frac{x^{\rho - s}}{(\rho - s)^2} d\sigma \Big).$$

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$$\Big| \sum_{\rho} \Big| \le \sum_{\rho} \frac{1}{|s_0 - \rho|^2} \int_{\sigma_0}^{\infty} x^{1/2 - \sigma} d\sigma = \frac{1}{e \log x} \sum_{\rho} \frac{1}{|s_0 - \rho|^2}$$

Hadamard's factorization formula + Stirling:

$$-\mathrm{Re}\, \frac{\zeta'}{\zeta}(s_0) = \frac{1}{2} \log T - \sum_{\rho} \mathrm{Re}\, \left(\frac{1}{s_0 - \rho} \right) = \frac{1}{2} \log T - \sum_{\rho} \frac{(\sigma_0 - 1/2)}{|s_0 - \rho|^2}.$$

Conclude:

$$\log|\zeta(s_0)| \leq \operatorname{Re} \sum_{n\leq x} \frac{\Lambda(n)}{n^{s_0}\log n}\frac{\log(x/n)}{\log x} + \frac{1}{2}\frac{\log T}{\log x}.$$

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Recall:

$$\log|\zeta(\tfrac{1}{2}+it)| \le \log|\zeta(s_0)| + \frac{1}{2}\frac{\log T}{\log x}.$$

Therefore

$$\log|\zeta(\tfrac{1}{2}+it)| \le \operatorname{Re} \sum_{n\le x} \frac{\Lambda(n)}{n^{s_0}\log n}\frac{\log(x/n)}{\log x} + \frac{\log T}{\log x}.$$

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Analogy with arithmetic functions:

$$\omega(n) \le \sum_{\substack{p|n\\ p\le y}} 1 + \frac{\log n}{\log y}.$$

Frequency of large values of $|\zeta(\frac{1}{2}+it)|$

Recall: $\mathcal{S}(T,V) = \{t \in [T,2T] : \log|\zeta(\frac{1}{2}+it)| \geq V\}$.

$$4 \leq A \leq \log_3 T; \qquad x = T^{A/V}; \qquad z = x^{1/\log\log T}$$

From Proposition: $\log|\zeta(\frac{1}{2}+it)| \leq S_1(t) + S_2(t) + V/A$ with

$$S_1(t) = \operatorname{Re} \sum_{p \leq z} \frac{1}{p^{\frac{1}{2}+it}}, \quad S_2(t) = \operatorname{Re} \sum_{z < p \leq x} \frac{1}{p^{\frac{1}{2}+it}}.$$

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Case 1: $S_1(t) \geq V(1-2/A)$.
Case 2: $S_2(t) \geq V/A$.

$$\mathrm{meas}(\mathcal{S}(T,V)) \leq \mathrm{meas}(\text{Case } 1) + \mathrm{meas}(\text{Case } 2).$$

Idea: bound meas(Case 1) and meas(Case 2) by computing moments of $S_1(t)$ and $S_2(t)$. $S_1(t)$ is more important, but by breaking at z, we can compute more moments of it.

Lemma: If $y^k \le T/\log T$ then

$$\int_T^{2T} \Big| \sum_{p \le y} \frac{a(p)}{p^{\frac{1}{2}+it}} \Big|^{2k} dt \ll Tk! \Big(\sum_{p \le y} \frac{|a(p)|^2}{p} \Big)^k.$$

Key point: Uniform in k.

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Proof:

$$\Big(\sum_{p \le y} \frac{a(p)}{p^{\frac{1}{2}+it}} \Big)^k = \sum_{n \le y^k} \frac{a_{k,y}(n)}{n^{\frac{1}{2}+it}}$$

If $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ then need all $p_i \le y$, $\sum_{\alpha_i} = k$ and then

$$a_{k,y}(n) = \binom{k}{\alpha_1, \dots, \alpha_r} \prod a(p_i)^{\alpha_i} .$$

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Since $y^k \leq T/\log T$, only diagonal terms matter:

$$\int_T^{2T} \Big| \sum_{n \leq y^k} \frac{a_{k,y}(n)}{n^{\frac{1}{2}+it}} \Big|^2 dt \ll T \sum_{n \leq y^k} \frac{|a_{k,y}(n)|^2}{n} \ll Tk! \Big(\sum_{p \leq y} \frac{|a(p)|^2}{p} \Big)^k.$$

Handling Case 1

Recall $z = T^{A/(V \log\log T)}$. For $k \le V \log\log T/(2A)$, Lemma gives

$$\int_T^{2T} |S_1(t)|^{2k} dt \ll Tk! \Big(\sum_{p \le z} \frac{1}{p} \Big)^k \ll T\sqrt{k} \Big(\frac{k \log\log T}{e} \Big)^k.$$

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Conclude:

$$\begin{aligned} \mathrm{meas}\{S_1(t) \ge V(1-2/A)\} &\ll (V(1-2/A))^{-2k} \int_T^{2T} |S_1(t)|^{2k} dt \\ &\ll T\sqrt{k}\Big(\frac{k \log\log T}{eV^2(1-2/A)^2}\Big)^k. \end{aligned}$$

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Important range: $\sqrt{\log\log T} \le V \le (\log\log T)^{\frac{3}{2}}$ choose $k = V^2/\log\log T$:

$$\mathrm{meas}\{S_1(t) \ge V(1-2/A)\} \ll \frac{TV}{\sqrt{\log\log T}}\exp\Big(-\frac{V^2}{(1-2/A)^2\log\log T}\Big)$$

In the range $V \geq (\log\log T)^{\frac{3}{2}}$ choose $k = 10V$ to get

$$\text{meas}\{S_1(t) \geq V(1 - 2/A)\} \ll T\exp(-V\log V).$$

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In either case conclude

$$\begin{aligned}\text{meas}&\{S_1(t) \geq V(1-2/A)\}\\ &\ll \frac{TV}{\sqrt{\log\log T}}\exp\Big(-\frac{V^2}{(1-2/A)^2\log\log T}\Big)\\ &\qquad + T\exp(-V\log V).\end{aligned}$$

First term is in the important range for moments — extrapolates Selberg's theorem in the large deviations range.

Handling Case 2

Recall $x = T^{A/V}$, $z = x^{1/\log\log T}$. For $k \le V/(2A)$, Lemma gives

$$\int_T^{2T} |S_2(t)|^{2k} dt \ll Tk! \Big(\sum_{z<p\le x} \frac{1}{p} \Big)^k \ll T(k \log_3 T)^k.$$

Conclude:

$$\operatorname{meas}\{S_2(t) \ge V/A\} \ll T \Big(\frac{k \log_3 T}{V^2/A^2} \Big)^k$$

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Conclude:

$$\operatorname{meas}\{S_2(t) \geq V/A\} \ll T\Big(\frac{k \log_3 T}{V^2/A^2}\Big)^k$$

Choose $k = V/2A$. Since $A \leq \log_3 T$ and we may assume $V \geq \sqrt{\log\log T}$, obtain

$$\operatorname{meas}\{S_2(t) \geq V/A\} \ll T \exp\Big(-\frac{V}{4A} \log V \Big).$$

Putting Case 1 and Case 2 together:

$$\begin{aligned}
&\operatorname{meas}(\mathcal{S}(T,V)) \\
&\qquad \ll \frac{TV}{\sqrt{\log\log T}} \exp\Big(-\frac{V^2}{(1-2/A)^2 \log\log T}\Big) \\
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\end{aligned}$$

Choose

$$A = \begin{cases} \log_3 T & \text{if } V \le \log_2 T \log_3 T \\ 4 & \text{if } V > \log_2 T \log_3 T \end{cases}$$

to deduce:

Theorem: If $\sqrt{\log_2 T} \le V = o(\log_2 T \log_3 T)$ then

$$\operatorname{meas}(\mathcal{S}(T,V)) \ll T\exp(-(1+o(1))V^2/\log\log T),$$

and for larger V it is $\ll T\exp(-cV\log V)$ for some $c > 0$.

Corollary:

$$\begin{aligned}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k} dt &= -\int_{-\infty}^{\infty} e^{2kV} d\mathrm{meas}(\mathcal{S}(T,V)) \\ &= 2k\int_{-\infty}^{\infty} e^{2kV} \mathrm{meas}\mathcal{S}(T,V) dV \\ &\ll T(\log T)^{k^2+\epsilon}.\end{aligned}$$

The main contribution coming from terms $V \approx k \log\log T$.

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$$\ll T(\log T)^{k^2+\epsilon}.$$

The main contribution coming from terms $V \approx k \log\log T$.
Related results: Jutila for $V \le \log\log T$

$$\mathrm{meas}(\mathcal{S}(T,V)) \ll T \exp\Big(-\frac{V^2}{\log\log T}\Big(1+O\Big(\frac{V}{\log\log T}\Big)\Big)\Big).$$

S. – resonance method – $V \le c\sqrt{\log T/\log_2 T}$

$$\mathrm{meas}(\mathcal{S}(T,V)) \gg \frac{T}{(\log T)^4} \exp\Big(-10\frac{V^2}{\log(\log T/(8V^2\log V))}\Big).$$

Harper's refinement

Iterative scheme similar to the argument for unconditional upper bounds.

Harper's choice of parameters:

$$\beta_0 = 0, \quad \beta_1 = \frac{1}{(\log\log T)^2}, \quad \beta_j = \frac{(20)^{j-1}}{(\log\log T)^2},$$

stop when $\beta_R \approx e^{-1000k}$.

Associated Dirichlet polynomials over primes: for $1 \le j \le R$

$$\mathcal{P}_j(t) = \sum_{T^{\beta_{j-1}} \le p \le T^{\beta_j}} \frac{1}{p^{\frac{1}{2}+it}}.$$

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$$\mathcal{P}_j(t) = \sum_{T^{\beta_{j-1}} \le p \le T^{\beta_j}} \frac{1}{p^{\frac{1}{2}+it}}.$$

Proposition gives for any $1 \le r \le R$:

$$\log|\zeta(\tfrac{1}{2}+it)| \le \mathrm{Re} \sum_{p \le T^{\beta_r}} \frac{1}{p^{\frac{1}{2}+it}} + \frac{1}{\beta_r}.$$

Key feature: flexibility in choosing which r to use.

Case 0: Suppose t is such that $|\mathrm{Re}\ \mathcal{P}_1(t)| \geq \beta_1^{-\frac{3}{4}} = (\log\log T)^{\frac{3}{2}}$.
Call the set of such $t \in [T, 2T]$ as $\mathcal{T}_0$.
For any $\ell \in \mathbb{N}$

$$\mathrm{meas}(\mathcal{T}_0)(\log\log T)^{3\ell} \leq \int_T^{2T} |\mathcal{P}_1(t)|^{2\ell} dt \ll T\ell!(\log\log T)^{\ell}.$$

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So

$$\int_{\mathcal{T}_0} |\zeta(\tfrac{1}{2}+it)|^{2k} dt \ll \Big(\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{4k} dt\Big)^{\frac{1}{2}} \Big(\mathrm{meas}(\mathcal{T}_0)\Big)^{\frac{1}{2}} \ll T.$$

Case r: For each $1 \leq j \leq r$ we have

$$|\mathrm{Re}\ \mathcal{P}_j(t)| \leq \beta_j^{-\frac{3}{4}},$$

but

$$|\mathrm{Re}\ \mathcal{P}_{r+1}(t)| > \beta_{r+1}^{-\frac{3}{4}}.$$

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Call this set $\mathcal{T}_r$.
Apply Proposition with this value r:

$$|\zeta(\tfrac{1}{2} + it)|^{2k} \ll \exp\Big(2k\text{Re } (\mathcal{P}_1(t) + \ldots + \mathcal{P}_r(t)) + \frac{2k}{\beta_r}\Big).$$

On $\mathcal{T}_r$ can bound (for each $1 \leq j \leq r$)

$$\exp(2k\text{Re } \mathcal{P}_j(t)) \ll E_{\ell_j}(2k\text{Re } \mathcal{P}_j(t)), \qquad \ell_j = 2\lfloor 100k\beta_j^{-\frac{3}{4}} \rfloor.$$

$$\int_{\mathcal{T}_r} |\zeta(\tfrac{1}{2}+it)|^{2k} dt \ll e^{2k/\beta_r} \int_{\mathcal{T}_r} \prod_{j=1}^{r} E_{\ell_j}(2k\mathrm{Re}\ \mathcal{P}_j(t)) dt$$

$$\ll e^{2k/\beta_r} \int_T^{2T} \prod_{j=1}^{r} E_{\ell_j}(2k\mathrm{Re}\ \mathcal{P}_j(t)) \Big(\beta_{r+1}^{\frac{3}{4}} |\mathcal{P}_{r+1}(t)|\Big)^{2\lfloor 1/(10\beta_{r+1})\rfloor} dt.$$

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Key points:

$$\prod_{j=1}^{r} E_{\ell_j}(2k\text{Re } \mathcal{P}_j(t)) \Big(\beta_{r+1}^{\frac{3}{4}} |\mathcal{P}_{r+1}(t)| \Big)^{2\lfloor 1/(10\beta_{r+1}) \rfloor}$$

is a short Dirichlet polynomial.

$$\text{Length } \ll T \text{ to the } \sum_{j=1}^{r} \ell_j \beta_j + \beta_{r+1}(2\lfloor 1/(10\beta_{r+1}) \rfloor) \ll T^{1/4}.$$

$\mathcal{P}_{r+1}(t)$ is rarely big – gain a lot from that term, enough to compensate e^{2k/β_r}.

Extreme values

Recall: Littlewood on GRH

$$\zeta(2)((2+o(1))e^{\gamma}\log\log|d|)^{-1} \le L(1,\chi_d) \le (2+o(1))e^{\gamma}\log\log|d|.$$

Know unconditionally (Chowla, ..., Granville & S.) there are arbitrarily large discriminants with

$$L(1,\chi_d) \ge e^{\gamma}(\log\log|d| + \log_3|d| - \log_4|d| + O(1)).$$

Correspondingly for small values of $L(1,\chi_d)$.
Conjecture (Granville & S.):

$$L(1,\chi_d) \le e^{\gamma}(\log\log|d| + \log_3|d| + C_1 + o(1)).$$

Related question: Least quadratic non-residue $\pmod p$.
Can get as large as $\gg \log p \log_3 p$ – Graham & Ringrose. On GRH gets as large as $\gg \log p \log\log p$ – Montgomery.
GRH implies that the least quadratic non-residue is $\le (\log p)^2$.

Story on the critical line?

Maximal size of $|\zeta(\frac{1}{2}+it)|$? Analogues for central values of L-functions?
Unconditional results: Subconvexity problem for L-functions.
Bourgain: $|\zeta(\frac{1}{2}+it)| \ll |t|^{13/84+\epsilon}$.

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On RH: Littlewood, ..., Chandee & S., Carneiro & Chandee

$$|\zeta(\tfrac{1}{2}+it)| \ll \exp\left(\left(\frac{\log 2}{2}+o(1)\right)\frac{\log|t|}{\log\log|t|}\right)$$

$$|\zeta(\sigma+it)| \ll \exp\left(C(\sigma)\frac{(\log|t|)^{2-2\sigma}}{\log\log|t|}\right)$$

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Explicit versions of slightly weaker result for L-functions: Chandee.
Questions: How large can we make $\zeta(\frac{1}{2}+it)$? Central values of L-functions? (Ω results)
What should be the truth?

Riemann implies Lindelöf

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) = e^{Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}$$

$$\left|\frac{\xi(\frac{1}{2}+it)}{\xi(-\frac{3}{2}+it)}\right| = \prod_{\rho}\left|\frac{i(\gamma-t)}{2+i(\gamma-t)}\right| = \prod_{\rho}\left|\frac{(t-\gamma)^2}{4+(t-\gamma)^2}\right|^{\frac{1}{2}}.$$

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Functional equation + Stirling: Put $f(x) = \log\frac{4+x^2}{x^2}$

$$\log|\zeta(\tfrac{1}{2}+it)| = \log t + O(1) - \frac{1}{2}\sum_{\gamma} f(t-\gamma)$$

Note:

$$\frac{1}{2}\int_{-\infty}^{\infty}\log\frac{4+x^2}{x^2}\,dx = 2\pi.$$

Size of $\zeta(\frac{1}{2}+it)$ related to fluctuations in the distribution of ordinates of zeros of $\zeta(s)$ (i.e. $S(t)$).

Note $f(x) = \log((4 + x^2)/x^2)$ has a singularity at $x = 0$.
Idea: Find a nice function g_Δ with $g_\Delta(x) \leq f(x)$, and such that

$$\widehat{g}_\Delta(\xi) = \int_{-\infty}^{\infty} g_\Delta(x) e^{-2\pi i x \xi} dx$$

is compactly supported in $[-\Delta, \Delta]$.

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$$\widehat{g}_\Delta(\xi) = \int_{-\infty}^{\infty} g_\Delta(x) e^{-2\pi i x\xi} dx$$

is compactly supported in $[-\Delta, \Delta]$. Then

$$\sum_\gamma f(t-\gamma) \ge \sum_\gamma g_\Delta(t-\gamma)$$

Now use explicit formula to convert RHS to a sum over primes.

$$\sum_\gamma g_\delta(t-\gamma) \approx \frac{\log t}{2\pi} \int_{-\infty}^{\infty} g_\Delta(u) - \frac{1}{\pi} \mathrm{Re} \sum_n \frac{\Lambda(n)}{n^{\frac{1}{2}+it}} \widehat{g}_\Delta\Big(\frac{\log n}{2\pi}\Big).$$

Conclude

$$\log|\zeta(\tfrac{1}{2}+it)| \le \frac{\log t}{4\pi} \int_{-\infty}^{\infty} (f(u) - g_\Delta(u)) du + \frac{1}{2\pi} \mathrm{Re} \sum_n \frac{\Lambda(n)}{n^{\frac{1}{2}+it}} \widehat{g}_\Delta\Big(\frac{\log n}{2\pi}\Big).$$

Estimate sum over primes trivially: Assuming $\widehat{g}_\Delta$ nice

$$\frac{1}{2\pi}\mathrm{Re}\sum_n \frac{\Lambda(n)}{n^{\frac{1}{2}+it}}\widehat{g}_\Delta\left(\frac{\log n}{2\pi}\right) \ll \sum_{n\le e^{2\pi\Delta}} \frac{\Lambda(n)}{\sqrt{n}} \ll e^{\pi\Delta}.$$

Problem: Find minorants $g_\Delta(x) \le f(x)$ with $\widehat{g}_\Delta$ supported in $[-\Delta, \Delta]$ with minimal

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Answered by work of Carneiro & Vaaler: minimum L^1 distance

$$= \frac{1}{\Delta}(2\log 2 - 2\log(1 + e^{-4\pi\Delta})).$$

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With this choice for g_Δ:

$$\log|\zeta(\tfrac{1}{2} + it)| \le \frac{\log t}{4\pi\Delta}(2\log 2 + O(e^{-4\pi\Delta})) + O(e^{\pi\Delta}).$$

Optimal: $\pi\Delta = (1-\epsilon)\log\log t$ gives bound for $\log|\zeta(\frac{1}{2} + it)|$.

Parallels the work of Goldston & Gonek for bounding $S(t)$: On RH

$$|S(t+h) - S(t)| \le \left(\frac{1}{2} + o(1)\right)\frac{\log t}{\log\log t}.$$

Multiplicity of a zero $\frac{1}{2} + i\gamma$ is bounded by $\log\gamma/(2\log\log\gamma)$.

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Refinement of Carneiro, Chandee, & Milinovich:

$$|S(t)| \le \left(\frac{1}{4} + o(1)\right) \frac{\log t}{\log \log t}.$$

Based on finding approximations to $(\arctan(1/x) - x/(1+x^2))$ — Carneiro & Littman.

Carneiro–Vaaler on polynomials

Suppose

$$F(z) = \prod_{n=1}^{N} (z - \alpha_n)$$

is a polynomial with all roots in the unit disc: $|\alpha_n| \leq 1$.

Theorem: For any integer M

$$\max_{|z|\leq 1} \log |F(z)| \leq \frac{\log 2}{M+1} N + \sum_{m=1}^{M} \frac{1}{m} \Big| \sum_{n=1}^{N} \alpha_n^m \Big|.$$

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Potential example: $N = rk$

$$F(z) = \prod_{j=1}^{r} (z - e(j/r))^k.$$

Analogue of $S(t)$: bounded by $k/2$.
Analogue of $\log|\zeta|$: bounded by 2^k.
Think of k as $\log t/(2 \log\log t)$.

Ω-results

Theorem: (Titchmarsh, Levinson)There are arbitrarily large t with (for $\frac{1}{2} \le \sigma \le 1$)

$$|\zeta(\sigma + it)| \ge \exp\left(A\frac{(\log t)^{1-\sigma}}{\log\log t}\right).$$

Note:

$$\prod_{p \le (\log t)} \left(1 - \frac{1}{p^\sigma}\right)^{-1} = \exp\left(B\frac{(\log t)^{1-\sigma}}{\log\log t}\right).$$

Recall RH upper bound: $\exp(C(\sigma)(\log t)^{2-2\sigma}/\log\log t)$.

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Recall RH upper bound: $\exp(C(\sigma)(\log t)^{2-2\sigma} / \log\log t)$.

Theorem: (Montgomery) Fix $1 \geq \sigma > \frac{1}{2}$. There are arbitrarily large t with

$$|\zeta(\sigma + it)| \geq \exp\left(\frac{\sqrt{\sigma - 1/2}}{20}\frac{(\log t)^{1-\sigma}}{(\log\log t)^\sigma}\right).$$

On RH:

$$|\zeta(\tfrac{1}{2} + it)| \geq \exp\left(\frac{1}{20}\frac{\sqrt{\log t}}{\sqrt{\log\log t}}\right).$$

Montgomery's approach: Use zero density estimates to

$$\text{approximate } \log|\zeta(\sigma+it)| \text{ by } \mathrm{Re} \sum_{p \le z} \frac{1}{p^{\sigma+it}}.$$

Here $z = c \log T \log\log T$. Use pigeonhole principle to find t with $cos(t \log p) \in (1/2, 1]$ for all $p \le z$.

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Alternative approach: Balasubramanian & Ramachandra.

$$\max_{T\le t\le T+H} |\zeta(\sigma+it)| \ge \left(\frac{1}{H}\int_T^{T+H} |\zeta(\sigma+it)|^{2k} dt\right)^{\frac{1}{2k}} \ge \left(\sum_{n\le H} \frac{d_k(n)^2}{n^{2\sigma}}\right)^{\frac{1}{2k}}.$$

Choose k to maximize. For $\sigma > 1/2$, only gives Levinson's result.

$$\max_{T\le t\le 2T} |\zeta(\tfrac{1}{2}+it)| \gg \exp\left(B\frac{\sqrt{\log T}}{\sqrt{\log\log T}}\right), \quad B = 0.53\ldots$$

True maximal order?

Montgomery suggested that these Ω results are optimal.
Seems solid for $\sigma > \frac{1}{2}$, but need more care on the critical line.

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Extrapolate Selberg's theorem: Perhaps measure of $t \in [T, 2t]$ with $\log|\zeta(\frac{1}{2}+it)| \geq V$ behaves something like

$$\asymp T \int_{V/\sqrt{\frac{1}{2}\log\log T}}^{\infty} e^{-x^2/2} dx \approx T \exp\left(-\frac{V^2}{\log\log T}\right).$$

Suggests maximal size for V about $\sqrt{\log T \log\log T}$.

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Suggests maximal size for V about $\sqrt{\log T \log\log T}$.
Conjecture: Farmer, Gonek & Hughes (2007) "Sub-Gaussian extreme values"

$$\max_{T \leq t \leq 2T} |\zeta(\tfrac{1}{2}+it)| = \exp\bigg(\Big(\frac{1}{\sqrt{2}} + o(1)\Big)\sqrt{\log T \log\log T}\bigg).$$

The moment conjectures: a paradox

Keating–Snaith conjecture

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k} dt \sim g_k \sum_{n\le T} \frac{d_k(n)^2}{n} \sim g_k a_k (\log T)^{k^2}.$$

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Asymptotics:

$$a_k \sim \left(\frac{e^{1-\gamma}}{2k^2\log k} + o(1)\right)^{k^2}; \quad g_k = \left(\frac{k}{4\sqrt{e}} + o(1)\right)^{k^2}.$$

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \quad \text{grows like } \left(\frac{C\log T}{k\log k}\right)^{k^2}.$$

The moment conjectures: a paradox

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Weak version: Uniformly in k one has

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$$\max_{T\le t\le 2T} |\zeta(\tfrac12+it)|^{2k}dt \ge \exp\left(\frac{c\log T}{\log\log T}\right).$$

CFKRS conjecture:

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \approx \int_0^T P_k\Big(\log\frac{t}{2\pi}\Big)\,dt$$

for a specified polynomial P_k of degree k^2.
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But this implies

$$\max_{T\le t\le 2T} |\zeta(\tfrac{1}{2}+it)| \le \min_k \Big(T(\log T)^{k^2}\Big)^{\frac{1}{2k}} = \exp\Big(\sqrt{\log T \log\log T}\Big).$$

Extreme values of L-functions

Earlier methods don't extend easily to central values.

Heath-Brown (unpublished), Hoffstein & Lockhart: There are arbitrarily large fundamental discriminants d such that

$$L(\tfrac{1}{2}, \chi_d) \gg \exp\left(C \frac{\sqrt{\log |d|}}{\log\log |d|}\right),$$

$$L(\tfrac{1}{2}, f \times \chi_d) \gg \exp\left(C \frac{\sqrt{\log |d|}}{\log\log |d|}\right).$$

Idea: Crucial use of quadratic reciprocity.

Can make $\chi_d(p) = \epsilon_p$ for all $p \leq z$, by choosing d in a progression $\pmod{4 \prod_{p \leq z} p}$.

Average over such progressions.

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Similar results of $|L(\frac{1}{2}, \chi)|$, $\chi \pmod{q}$? or $L(\frac{1}{2}, f)$ as f ranges over all Hecke eigenforms of large weight k (or large level)?

One motivation for work with Rudnick on lower bounds for moments in families.

The resonance method

Idea: Resonator – $R(t) = \sum_n r(n)n^{-it}$. Compute

$$I_1 = \int_T^{2T} |R(t)|^2 dt, \qquad I_2 = \int_T^{2T} \zeta(\tfrac{1}{2} + it)|R(t)|^2 dt.$$

Then

$$\max_{T \le t \le 2T} |\zeta(\tfrac{1}{2} + it)| \ge \frac{|I_2|}{I_1}.$$

Problem: Choose $a(n)$ so as to maximize the ratio $|I_2|/I_1$.

If $R(t)$ is a short Dirichlet polynomial then both I_1 and I_2 can be evaluated. Two quadratic forms in the coefficients $r(n)$ – problem is to optimize their ratio.

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$$\max_{T \le t \le 2T} |\zeta(\tfrac{1}{2} + it)| \ge \frac{|I_2|}{I_1}.$$

Problem: Choose $a(n)$ so as to maximize the ratio $|I_2|/I_1$.

If $R(t)$ is a short Dirichlet polynomial then both I_1 and I_2 can be evaluated. Two quadratic forms in the coefficients $r(n)$ – problem is to optimize their ratio.

Method widely applicable:

$$\sum_{\chi \pmod q} L(\tfrac{1}{2}, \chi)|R(\chi)|^2 \Big/ \sum_{\chi \pmod q} |R(\chi)|^2,$$

$$\sideset{}{^\flat}\sum_{|d| \le X} L(\tfrac{1}{2}, \chi_d)|R(d)|^2 \Big/ \sideset{}{^\flat}\sum_{|d| \le X} |R(d)|^2$$

The quadratic forms

$$R(t) = \sum_{n \le N} r(n) n^{-it}$$

If $N \le T^{1-\epsilon}$ – short Dirichlet polynomial – only diagonal terms matter.

$$I_1 = \int_T^{2T} |R(t)|^2 dt \sim T \sum_{n \le N} |r(n)|^2$$

$$I_2 \approx \int_T^{2T} \sum_{k \le T} \frac{1}{k^{\frac{1}{2}+it}} \sum_{m,n \le N} r(m)\overline{r(n)} \left(\frac{n}{m}\right)^{it} dt \approx T \sum_{mk=n \le N} \frac{r(m)\overline{r(n)}}{\sqrt{k}}.$$

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Similar quadratic forms in other families: e.g. $N \le \sqrt{q}$,

$$\frac{1}{\phi(q)} \sum_{\chi \pmod q} |R(\chi)|^2 \approx \sum_{n \le N} |r(n)|^2,$$

$$\frac{1}{\phi(q)} \sum_{\chi \pmod q} L(\tfrac{1}{2}, \chi) |R(\chi)|^2 \approx \sum_{km=n \le N} \frac{r(m)\overline{r(n)}}{\sqrt{k}}$$

Or, with $N \le X^{\frac{1}{4}}$

$$\sum_{|d|\le X}^{\flat} R(\chi_d)^2 \sim CX \sum_{\substack{n_1,n_2\le N\\ n_1n_2=\square}} r(n_1)r(n_2)$$

$$\sum_{|d|\le X}^{\flat} L(\tfrac{1}{2},\chi_d)R(\chi_d)^2 \sim CX \sum_{\substack{k\le\sqrt{X}\\ n_1,n_2\le N\\ kn_1n_2=\square}} \frac{r(n_1)r(n_2)}{\sqrt{k}}.$$

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Theorem: (S.) For large N

$$\max_r \Big| \sum_{mk \le N} \frac{r(m)\overline{r(mk)}}{\sqrt{k}} \Big| \Big/ \Big(\sum_{n \le N} |r(n)|^2 \Big) = \exp\left((1+o(1)) \frac{\sqrt{\log N}}{\sqrt{\log \log N}} \right).$$

Corollary: There exist $t \in [T, 2T]$, $|d| \in [X, 2X]$ with

$$|\zeta(\tfrac{1}{2} + it)| \ge \exp\left((1 + o(1)) \frac{\sqrt{\log T}}{\sqrt{\log \log T}} \right),$$

$$L(\tfrac{1}{2}, \chi_d) \ge \exp\left(\frac{1}{3} \frac{\sqrt{\log X}}{\sqrt{\log \log X}} \right).$$

For all $3 \leq V \leq \frac{1}{5}\sqrt{\log T/\log\log T}$ the measure of $t \in [T, 2T]$ with $|\zeta(\frac{1}{2}+it)| \geq e^V$ exceeds

$$\frac{T}{(\log T)^4} \exp\Big(-10 \frac{V^2}{\log(\log T/(8V^2 \log V))}\Big).$$

The set on which $|\zeta(\frac{1}{2}+it)| \geq \exp(\frac{1}{5}\sqrt{\log T/\log\log T})$ has measure

$$\gg T \exp\Big(-c \frac{\log T}{\log\log T}\Big),$$

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Can use a similar argument to produce large values of $|\zeta(\sigma+it)|$, $L(\sigma, \chi_d)$ etc.

Hilberdink; Voronin

Like with moments, this only gives Levinson type Ω–results: e.g.

$$|\zeta(\sigma+it)| \gg \exp\Big(C(\sigma)\frac{(\log t)^{1-\sigma}}{\log\log t}\Big).$$

Optimizing the ratio of quadratic forms

Theorem: (S.) For large N

$$\max_r \Big| \sum_{mk \le N} \frac{r(m)\overline{r(mk)}}{\sqrt{k}} \Big| \Big/ \Big(\sum_{n \le N} |r(n)|^2 \Big) = \exp\Big((1+o(1)) \frac{\sqrt{\log N}}{\sqrt{\log\log N}} \Big).$$

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The lower bound in the theorem.

Choose $r(n) = f(n)$ — real valued multiplicative function supported on square-free numbers.

Denominator:

$$\sum_{n \le N} |r(n)|^2 \le \prod_p (1 + f(p)^2).$$

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Numerator:

$$\sum_{k\le N} \frac{f(k)}{\sqrt{k}} \sum_{\substack{m\le N/k \\ (m,k)=1}} f(m)^2$$

Rankin's trick, for any $\alpha > 0$: Numerator

$$\geq \sum_k \frac{f(k)}{\sqrt{k}} \sum_{(m,k)=1} f(m)^2 - \frac{1}{N^\alpha} \sum_k \frac{f(k)}{k^{\frac{1}{2}-\alpha}} \sum_{(m,k)=1} f(m)^2 m^\alpha$$

$$= \prod_p \left(1 + \frac{f(p)}{\sqrt{p}} + f(p)^2\right) - \frac{1}{N^\alpha} \prod_p \left(1 + p^\alpha \left(\frac{f(p)}{\sqrt{p}} + f(p)^2\right)\right).$$

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Heuristic choice of resonator: Suppose $f(p) \leq 1$ always, that $f(p)^2$ dominates $f(p)/\sqrt{p}$, and that $p^\alpha - 1 \approx \alpha \log p$.
Then

$$\text{Numerator} \geq \frac{1}{2} \prod_p \left(1 + \frac{f(p)}{\sqrt{p}} + f(p)^2\right),$$

provided constraint:

$$\sum_p f(p)^2 \log p \leq \log N - (\log 2)/\alpha.$$

Problem: Maximize ratio

$$\prod_p \left(1 + \frac{f(p)}{\sqrt{p}} + f(p)^2\right) \Big/ \left(1 + f(p)^2\right) \approx \exp\left(\sum_p \frac{f(p)}{\sqrt{p}}\right)$$

under constraint

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Motivates the choice: $L = \sqrt{\log N \log\log N}$, $\alpha = 1/(\log L)^3$

$$f(p) = \frac{L}{\sqrt{p}\log p} \quad \text{for } L^2 \le p \le \exp((\log L)^2).$$

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$$f(p)=\frac{L}{\sqrt{p}\log p} \quad \text{for } L^2 \le p \le \exp((\log L)^2).$$

$$\sum_p f(p)^2 \log p = \sum_{L^2\le p\le \exp((\log L)^2)} \frac{L^2}{p\log p} \approx \frac{L^2}{\log L} < \log N$$

$$\sum_p \frac{f(p)}{\sqrt{p}} = \sum_{L^2\le p\le \exp((\log L)^2)} \frac{L}{p\log p} \approx \frac{L}{\log L} \approx \frac{\sqrt{\log N}}{\sqrt{\log\log N}}.$$

The upper bound of the Theorem

Guess that the lower bound example is close to optimal.
Define g multiplicative by $(L = \sqrt{\log N \log\log N})$

$$g(p^k) = \min\left(1, \frac{L}{p^{k/2}\log p}\right).$$

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$$r(mk)r(m) \le \frac{1}{2}\left(\frac{r(mk)^2}{g(k)} + g(k)r(m)^2\right)$$

$$\begin{aligned}\sum_{km\le N} \frac{r(m)r(mk)}{\sqrt{k}} &\le \frac{1}{2}\sum_{km\le N}\frac{1}{\sqrt{k}}\left(\frac{r(mk)^2}{g(k)} + g(k)r(m)^2\right)\\ &= \frac{1}{2}\sum_{n\le N} r(n)^2\left(\sum_{k|n}\frac{1}{\sqrt{k}g(k)} + \sum_{k\le N/n}\frac{g(k)}{\sqrt{k}}\right).\end{aligned}$$

$$\text{Ratio} \le \frac{1}{2}\max_{n\le N}\sum_{k|n}\frac{1}{\sqrt{k}g(k)} + \frac{1}{2}\sum_{k\le N}\frac{g(k)}{\sqrt{k}}$$

$$\begin{aligned}
\sum_{k \le N} \frac{g(k)}{\sqrt{k}} &\le \prod_p \Big(1 + \frac{g(p)}{\sqrt{p}} + \frac{g(p)^2}{p} + \dots\Big) \\
&\ll \exp\Big(\sum_{p \le \log N / \log\log N} \frac{1}{\sqrt{p}-1} + 2 \sum_{p > \log N / \log\log N} \frac{L}{p \log p}\Big) \\
&= \exp\Big((1+o(1))\frac{\sqrt{\log N}}{\sqrt{\log\log N}}\Big).
\end{aligned}$$

$$\sum_{k\le N}\frac{g(k)}{\sqrt{k}} \le \prod_p\left(1+\frac{g(p)}{\sqrt{p}}+\frac{g(p)^2}{p}+\dots\right)$$

$$\ll \exp\left(\sum_{p\le \log N/\log\log N}\frac{1}{\sqrt{p}-1}+2\sum_{p>\log N/\log\log N}\frac{L}{p\log p}\right)$$

$$=\exp\left((1+o(1))\frac{\sqrt{\log N}}{\sqrt{\log\log N}}\right).$$

Since $g(p^a)=\min(1, L/(p^{a/2}\log p))$,

$$\sum_{k|n}\frac{1}{\sqrt{k}g(k)} \le \prod_{p^a\|n}\left(1+\frac{a\log p}{L}\right)\prod_{p|n}\left(1+\frac{1}{\sqrt{p}-1}\right).$$

First factor $\le \exp\left(\sum_{p^a\|n}\frac{a\log p}{L}\right)\le n^{1/L}\le N^{1/L}=\exp\left(\frac{\sqrt{\log N}}{\sqrt{\log\log N}}\right)$.

Second factor $\le \exp\left(O\left(\sum_{p|n}\frac{1}{\sqrt{p}}\right)\right)\le \exp\left(O\left(\frac{\sqrt{\log N}}{\log\log N}\right)\right)$.

Frequency of large values

$$\Big|\int_T^{2T}\zeta(\tfrac12+it)|R(t)|^2dt\Big| \le e^V\int_T^{2T}|R(t)|^2dt + \int_{\mathcal{L}}|\zeta(\tfrac12+it)||R(t)|^2dt$$

where $\mathcal{L}$ is the subset of $[T,2T]$ on which $|\zeta(\frac12+it)|\ge e^V$.

Frequency of large values

$$\Big|\int_T^{2T} \zeta(\tfrac{1}{2}+it)|R(t)|^2 dt\Big| \le e^V \int_T^{2T} |R(t)|^2 dt + \int_{\mathcal{L}} |\zeta(\tfrac{1}{2}+it)||R(t)|^2 dt$$

where $\mathcal{L}$ is the subset of $[T, 2T]$ on which $|\zeta(\frac{1}{2}+it)| \ge e^V$.
Two applications of Cauchy–Schwarz —- second term

$$\le \Big(\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^4 dt\Big)^{\frac{1}{4}} \Big(\text{meas}(\mathcal{L})\Big)^{\frac{1}{4}} \Big(\int_T^{2T} |R(t)|^4 dt\Big)^{\frac{1}{2}}.$$

If $N \le T^{1/2-\epsilon}$ can compute 4-th moment of $R(t)$.
Small modifications to the resonator yield bounds for meas($\mathcal{L}$).