The maximum of the zeta function in intervals of length 1

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Fyodorov, Hiary, Keating problem

Choose t uniformly from $[T, 2T]$.

What is the distribution of

$$\max_{|t-u|\leq 1} \log |\zeta(\tfrac{1}{2} + iu)|?$$

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Natural problem from random matrix theory:
Pick a large random $g \in U(N)$. What is the distribution of

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Conjecture: Fyodorov, Hiary, Keating:

$$\max_{|t-u|\leq 1} \log |\zeta(\tfrac{1}{2} + iu)| = \log\log T - \frac{3}{4} \log_3 T + X_T,$$

for a random variable X_T whose distribution is explicitly given.

$X_T = O(1)$ almost surely.

What does this mean?

Spacing between zeros at height T is $\approx 2\pi/\log T$.
Roughly speaking: ζ changes on the scale of $1/\log T$.
Think of an interval of length 1 as having about $\log T$ different values of $\zeta(\frac{1}{2}+iu)$.

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Selberg's theorem: $\log|\zeta(\frac{1}{2}+it)|$ is normal with mean 0 and variance $\sim \frac{1}{2}\log\log T$.

First Guess: Pick $\log T$ independent samples of a Gaussian with mean zero and variance $\frac{1}{2}\log\log T$. What is the typical size of the maximum of these samples?

Toy problem

Pick N independent standard normal variables. What is the distribution of their maximum?

Probability that standard normal variable $\leq M$ is

$$1 - \frac{1}{\sqrt{2\pi}} \int_M^\infty e^{-x^2/2} dx \approx 1 - C \frac{e^{-M^2/2}}{M}.$$

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$$Me^{M^2/2} \approx N; \quad M \approx \sqrt{2 \log N}$$

$$e^{M^2/2} \approx \frac{N}{\sqrt{\log N}}; \quad \frac{M^2}{2} = \log N - \tfrac{1}{2} \log\log N$$

$$M = \sqrt{2\log N}\left(1 - \frac{1}{4}\frac{\log_2 N}{\log N}\right).$$

Take $N = \log T$, and scale by $\sqrt{\frac{1}{2}\log\log T}$.
Suggests

$$\max_{|t-u|\leq 1} \log|\zeta(\tfrac{1}{2} + iu)| = \log\log T\left(1 - \frac{1}{4}\frac{\log_3 T}{\log_2 T}\right).$$

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Fyodorov, Hiary, Keating:

$$\max_{|t-u|\le 1} \log|\zeta(\tfrac{1}{2}+it)| = \log\log T\left(1-\frac{3}{4}\frac{\log_3 T}{\log_2 T}\right).$$

Why the discrepancy?

Take $N = \log T$, and scale by $\sqrt{\frac{1}{2}\log\log T}$.
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$$\max_{|t-u|\le 1} \log|\zeta(\tfrac{1}{2}+iu)| = \log\log T\left(1 - \frac{1}{4}\frac{\log_3 T}{\log_2 T}\right).$$

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$$\max_{|t-u|\le 1} \log|\zeta(\tfrac{1}{2}+it)| = \log\log T\left(1 - \frac{3}{4}\frac{\log_3 T}{\log_2 T}\right).$$

Why the discrepancy?
Answer: Values of $\log|\zeta(\frac{1}{2}+iu)|$ don't quite behave like $\log T$ independent Gaussians. Nearby values are correlated.

Correlations of nearby values of $\zeta(\frac{1}{2}+it)$

Covariance of $\log|\zeta(\frac{1}{2}+it)|$ and $\log|\zeta(\frac{1}{2}+it+ih)|$.
Think of prime sums:

$$\sum_{p\le x}\frac{1}{p^{\frac{1}{2}+it}},\ \text{and}\ \sum_{p\le x}\frac{1}{p^{\frac{1}{2}+it+ih}}.$$

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For the primes $p \le e^{1/h}$ we have $p^{it} \approx p^{it+ih}$.
Larger primes are uncorrelated.
Analogue of Selberg's theorem: Covariance of $\log|\zeta(\frac{1}{2}+it)|$ and $\log|\zeta(\frac{1}{2}+it+ih)|$ equals

$$\frac{1}{2}\sum_{p\le\min(e^{1/h},T)}\frac{1}{p}=\frac{1}{2}\log\min(h^{-1},\log T).$$

The picture

For each $k \leq \log_2 T$ consider

$$\mathcal{P}_k(u) = \mathrm{Re} \sum_{e^{e^k} \leq p < e^{e^{k+1}}} \frac{1}{p^{\frac{1}{2}+iu}}.$$

These behave independently, like Gaussians with mean 0 and variance $\frac{1}{2}$.

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Given t uniformly in $[T, 2T]$ and as u varies in $[t-1, t+1]$ how do these $\mathcal{P}_k(u)$ change?

Note: $\mathcal{P}_k$ changes on the scale of e^{-k}.

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Picture for $\log|\zeta(\frac{1}{2}+it)|$:

$$\mathcal{P}_1\Big(t + \frac{i_1}{e}\Big) + \mathcal{P}_2\Big(t + \frac{i_1}{e} + \frac{i_2}{e^2}\Big) + \mathcal{P}_3\Big(t + \frac{i_1}{e} + \frac{i_2}{e^2} + \frac{i_3}{e^3}\Big) + \dots.$$

where $i_k \le e^k$.

Branching Brownian motion

Start at time 0 and perform standard Brownian motion.

At time t there is a chance e^{-t} that the particle splits into two.

The two new particles both perform standard Brownian motion starting at this point.

After further time t they have a chance e^{-t} of splitting into two.

And so on.

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Theorem (Bramson): The maximum looks almost surely like

$$\sqrt{2}\left(T - \frac{3}{4}\log T\right) + O(1).$$

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Toy Problem: $X_{i,j} = \pm 1$ equal probability. Maximum of

$$X_0 + X_{1,i_1} + X_{2,i_2} + \ldots + X_{k,i_k}$$

where $1 \leq i_r \leq 2^r$ for each $1 \leq r \leq k$?

What is known?

Large literature in random matrix theory: Paquette & Zeitouni.
With probability 1 (for picking random matrix g)

$$\max_{\theta} \log |\det(e^{i\theta} I - g)| = \log N - \frac{3}{4} \log_2 N + O(1).$$

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Theorem: Arguin, Belius, Harper For each prime p let $X(p)$ denote independent random variables uniform on unit circle. Then

$$\max_{h \in [0,1]} \operatorname{Re} \sum_{p \le T} \frac{X(p)}{p^{\frac{1}{2}+ih}} = \log \log T - \frac{3}{4} \log_3 T + o(\log_3 T).$$

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Theorem: Arguin, Belius, Bourgade, Radziwill, & S.; Najnudel For almost all t

$$\max_{|t-u|\le 1} |\zeta(\tfrac{1}{2} + iu)| = (\log T)^{1+o(1)}.$$

Related work

Fyodorov, Hiary, Keating: Conjecture on moments of ζ in intervals of bounded length.

Theorem: Arguin, Ouimet, Radziwill If $\beta \le 2$ then for almost all t

$$\int_{-1}^{1} |\zeta(\tfrac{1}{2} + it + ih)|^{\beta} dh = (\log T)^{\frac{\beta^2}{4} + o(1)}.$$

If $\beta > 2$, then for almost all t

$$\int_{-1}^{1} |\zeta(\tfrac{1}{2} + it + ih)|^{\beta} dh = (\log T)^{\beta - 1 + o(1)}.$$

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Random multiplicative functions: the work of Harper

$X(p)$ independent random variables uniform on $\mathbb{T}$.

Extend completely multiplicatively to $X(n)$ – random multiplicative function.

What can one say about the distribution of

$$\sum_{n \leq x} X(n)?$$

$$\sum_{p \le x} X(p) \quad \text{Central Limit Theorem}$$

$$\sum_{\substack{n \le x \\ \omega(n) \le k}} X(n) \quad \text{Gaussian — Hough, Harper}$$

$$\sum_{x \le n \le x+y} X(n) \quad \text{Gaussian if } y = o(x/\log x) \text{ Chatterjee \& S.}$$

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$$\mathbb{E}\Big|\sum_{n \le x} X(n)\Big|^2 = x.$$

Theorem: Harper

$$\mathbb{E}\Big|\sum_{n \le x} X(n)\Big| \asymp \frac{\sqrt{x}}{(\log\log x)^{\frac{1}{4}}}.$$

Established Helson's conjecture: $\mathbb{E}|\sum_{n \le x} X(n)| = o(\sqrt{x})$.

Relevance to earlier problems:

$$\mathbb{E}\Big|\sum_{n\leq x} X(n)\Big| \asymp \sqrt{x}\,\mathbb{E}\Big[\Big(\frac{1}{\log x}\int_{-\frac{1}{2}}^{\frac{1}{2}} |F_X(\tfrac{1}{2}+it)|^2 dt\Big)^{\frac{1}{2}}\Big]$$

where

$$F_X(s) = \prod_{p\leq x}\Big(1-\frac{X(p)}{p^s}\Big)^{-1}.$$

Relevance to earlier problems:

$$\mathbb{E}\Big|\sum_{n\le x} X(n)\Big| \asymp \sqrt{x}\,\mathbb{E}\Big[\Big(\frac{1}{\log x}\int_{-\frac12}^{\frac12} |F_X(\tfrac12 + it)|^2 dt\Big)^{\frac12}\Big]$$

where

$$F_X(s) = \prod_{p\le x}\Big(1 - \frac{X(p)}{p^s}\Big)^{-1}.$$

Conjecture(?):

$$\frac{1}{T}\int_0^T \Big(\frac{1}{\log T}\int_0^1 |\zeta(\tfrac12 + it + ih)|^2 dh\Big)^{\frac12} dt \asymp \frac{1}{(\log\log T)^{\frac14}}.$$

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Theorem: S. & Zaman (in progress) $f(z) = \sum_{n=1}^{\infty} X(n) z^n/\sqrt{n}$ where $X(n)$ are independent standard complex Gaussians. Put

$$F(z) = \exp(f(z)) = \sum_{n=0}^{\infty} a(n) z^n.$$

Then, almost surely, $a(n) \to 0$ as $n \to \infty$. In fact:

$$\mathbb{E}(|a(n)|) \ll (\log n)^{-\frac{1}{4}}.$$

Ideas behind Arguin, Belius, Bourgade, Radziwill, & S.

Theorem: For almost all $t \in [T, 2T]$

$$\max_{|t-u|\le 1} |\zeta(\tfrac{1}{2} + iu)| = (\log T)^{1+o(1)}.$$

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Proof of the upper bound:
A Sobolev inequality:

$$f(u)^2 = \frac{f(1)^2 + f(-1)^2}{2} + \int_{-1}^{u} f'(v)f(v)dv - \int_{u}^{1} f'(v)f(v)dv$$

$$\max_{u\in[-1,1]} |f(u)|^2 \leq \frac{|f(1)|^2}{2} + \frac{|f(-1)|^2}{2} + \int_{-1}^{1} |f'(v)f(v)|dv.$$

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Conclude:

$$\max_{|t-u|\le 1} |\zeta(\tfrac{1}{2} + iu)|^2 \ll |\zeta(\tfrac{1}{2} + it \pm i)|^2 + \int_{-1}^{1} |\zeta'(\tfrac{1}{2} + it)\zeta(\tfrac{1}{2} + it)|dt.$$

Hence

$$\frac{1}{T}\int_T^{2T}\Big(\max_{|t-u|\le 1}|\zeta(\tfrac12+iu)|^2\Big)dt \ll \frac{1}{T}\int_T^{2T}\Big(|\zeta(\tfrac12+it)|^2+|\zeta'(\tfrac12+it)\zeta(\tfrac12+it)|\Big)dt.$$

Easy:

$$\int_T^{2T}|\zeta(\tfrac12+it)|^2dt \ll T(\log T);\quad \int_T^{2T}|\zeta'(\tfrac12+it)|^2dt \ll T(\log T)^3.$$

Hence

$$\frac{1}{T}\int_T^{2T}\Big(\max_{|t-u|\le 1}|\zeta(\tfrac12+iu)|^2\Big)\,dt \ll \frac{1}{T}\int_T^{2T}\Big(|\zeta(\tfrac12+it)|^2+|\zeta'(\tfrac12+it)\zeta(\tfrac12+it)|\Big)\,dt.$$

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By Cauchy–Schwarz

$$\frac{1}{T}\int_T^{2T}\Big(\max_{|t-u|\le 1}|\zeta(\tfrac12+iu)|^2\Big)\,dt \ll (\log T)^2.$$

$$\mathrm{meas}\Big(t\in[T,2T]:\max_{|t-u|\le 1}|\zeta(\tfrac12+iu)|>V\log T\Big)\ll \frac{T}{V^2}.$$

Ideas for the lower bound

First part: Convert to prime sums

A large value of $\zeta(\sigma + it)$ implies a large value of ζ near $1/2 + it$.

Lemma Suppose $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + (\log T)^{-\frac{1}{2}-\epsilon}$. Then

$$\mathbb{P}\Big(\max_{|t-u|\leq 1} |\zeta(\tfrac{1}{2}+iu)| \geq V\Big) \geq \mathbb{P}\Big(\max_{|t-u|\leq 1/4} |\zeta(\sigma+iu)| \geq 2V\Big) + o(1).$$

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K large integer. Put

$$\sigma_0 = \frac{1}{2} + \frac{(\log T)^{\frac{3}{2K}}}{\log T}; \quad X = \exp\left((\log T)^{1-\frac{1}{K}}\right)$$

Use mollifiers to prove: most of the time

$$\zeta(\sigma_0 + iu) \prod_{p \le X} \left(1 - \frac{1}{p^{\sigma_0 + iu}}\right) \approx 1.$$

In fact, this holds for all $u \in [t-1, t+1]$ for almost all t.

Reduced to understanding

$$\max_{|t-u|\leq 1} \mathrm{Re} \sum_{p\leq X} \frac{1}{p^{\sigma_0+iu}}.$$

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Split the prime sum into $K-1$ different ranges:

$$J_0 = [2, \exp((\log T)^{\frac{1}{K}})],$$

$$J_j = (\exp((\log T)^{\frac{j}{K}}), \exp((\log T)^{\frac{j+1}{K}})], \quad 1 \le j \le K-2.$$

$$\mathcal{P}_j(u) = \mathrm{Re} \sum_{p\in J_j} \frac{1}{p^{\sigma_0+iu}}.$$

Note: each $\mathcal{P}_j(u)$ for $0 \le j \le K-3$ is approximately Gaussian with mean 0 and variance

$$\sim \frac{1}{2} \sum_{p\in J_j} \frac{1}{p^{2\sigma_0}} \sim \frac{1}{2K} \log\log T.$$

Different $\mathcal{P}_j$ are uncorrelated.

$\mathcal{P}_j$ changes on the scale of $(\log T)^{-\frac{j}{K}}$.

Conclude:

$$\mathbb{P}\Big(\max_{|u-t|\leq 1} \log|\zeta(\tfrac{1}{2}+iu)| \geq (1-2\epsilon)\log\log T\Big)$$
$$\geq \mathbb{P}\Big(\max_{|u-t|\leq \frac{1}{4}} \sum_{j=1}^{K-3} P_j(u) \geq (1-\epsilon)\log\log T\Big) + o(1).$$

Conclude:

$$\mathbb{P}\Big(\max_{|u-t|\le 1}\log|\zeta(\tfrac12+iu)|\ge(1-2\epsilon)\log\log T\Big)$$
$$\ge\mathbb{P}\Big(\max_{|u-t|\le\frac14}\sum_{j=1}^{K-3}P_j(u)\ge(1-\epsilon)\log\log T\Big)+o(1).$$

Key step:

$$\mathbb{P}\Big(\max_{|u-t|\le\frac14}\Big(\mathcal{P}_j(u)\ge\frac{(1-\epsilon)}{K}\log\log T\ \text{ for all } 1\le j\le K-3\Big)\Big)=1+o(1).$$

Shouldn't just make $\sum\mathcal{P}_j$ large, but each constituent must be large!

Note:

$$\mathbb{P}(\mathcal{P}_j(u)>\frac{1}{K}\log\log T)\approx\exp\Big(-\frac{(\frac1K\log\log T)^2}{\frac1K\log\log T}\Big)=(\log T)^{-\frac1K}.$$

Idea behind the key step

Imagine $u = t + k/\log T$, and that $0 \le k < \log T$.
Let $\mathcal{T}(k)$ be the event: (with $\lambda < 1$)

$$P_j(t + k/\log T) \ge \frac{\lambda}{K} \log\log T, \quad \text{for all } 1 \le j \le K - 3.$$

This has probability about $(\log T)^{-\lambda^2(K-3)/K}$.

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By Cauchy–Schwarz

$$\mathbb{P}\Big(\bigcup_{0\le k<\log T} \mathcal{T}(k)\Big) \ge \Big(\sum_k \mathbb{P}(\mathcal{T}(k))\Big)^2 \Big/ \sum_{k,\ell} \mathbb{P}(\mathcal{T}(k) \cap \mathcal{T}(\ell))$$

Proof:

$$\Big(\mathbb{E}\Big[\sum_k \mathbf{1}_{\mathcal{T}(k)}\Big]\Big)^2 = \Big(\mathbb{E}\Big[\mathbf{1}_{\cup_k \mathcal{T}(k)} \sum_k \mathbf{1}_{\mathcal{T}(k)}\Big]\Big)^2$$
$$\le \mathcal{P}\Big(\bigcup_k \mathcal{T}(k)\Big)\mathbb{E}\Big[\Big(\sum_k \mathbf{1}_{\mathcal{T}(k)}\Big)^2\Big].$$

Numerator:

$$\Big(\sum_k \mathbb{P}(\mathcal{T}(k))\Big)^2 \asymp (\log T \times (\log T)^{-\lambda^2(K-3)/K})^2.$$

Goal: Show that denominator $\approx$ numerator.
Key: $\mathcal{P}_j$ changes on the scale of $(\log T)^{-\frac{j+1}{K}}$.
Typical case: k and ℓ are not close to each other:
$|k-\ell| \geq (\log T)^{1-1/2K}$.
Then all the $P_j(t+k/\log T)$ behave independently of
$P_j(t+\ell/\log T)$.
So

$$\mathbb{P}(\mathcal{T}(k) \cap \mathcal{T}(\ell)) \approx \mathbb{P}(T(k)) \times \mathbb{P}(\mathcal{T}(\ell)).$$

These terms give:

$$\approx \Big(\sum_k \mathbb{P}(T(k))\Big)\Big(\sum_\ell \mathbb{P}(\mathcal{T}(\ell))\Big).$$

Atypical case: $k - \ell \approx (\log T)^{1-\frac{r}{K}}$.
For $j \leq r$, $\mathcal{P}_j(t + k/\log T)$ and $\mathcal{P}_j(t + \ell/\log T)$ are strongly correlated.
But for $r + 1 \leq j \leq K - 3$ they behave independently.
Probability: $(\log T)^{-\lambda^2 r/K}(\log T)^{-2\lambda^2(K-3-r)/K}$.
Multiplied by number of atypical cases: $(\log T)^{2-r/K}$ gives the result.