Extreme values of zeta and L-functions: Recent advances

K. Soundararajan

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Refinements of the resonance method

Recent work: originating with (i) Aistleitner, (ii) Bondarenko & Seip.
Extensions by (iii) Aistleitner, Mahatab, Munsch, Peyrot; (iv) de la Breteche and Tenenbaum.
Related to progress on "Gál sums" or "gcd sums".
Theorem: (Bondarenko & Seip) There is a set $\mathcal{N} = \{n_1, \ldots, n_N\}$ of N natural numbers such that

$$\sum_{k,\ell=1}^{N} \frac{(n_k, n_\ell)}{\sqrt{n_k n_\ell}} \geq N \exp\left((1-\epsilon)\frac{\sqrt{\log N \log_3 N}}{\sqrt{\log_2 N}}\right).$$

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Allow for resonators with larger terms.
Exploit positivity — known at present only for $\zeta(s)$, and $L(\sigma, \chi)$ for characters $\chi \pmod q$.

Large values $\sigma = 1/2$

Theorem: (Bondarenko & Seip) For all large T

$$\max_{\sqrt{T} \le t \le T} |\zeta(\tfrac{1}{2} + it)| \ge \exp\left((1-\epsilon)\frac{\sqrt{\log T \log_3 T}}{\sqrt{\log_2 T}}\right).$$

Note localization to $[\sqrt{T}, T]$. Method does not give large values on $[T, 2T]$. Also crucial that the coefficients of $\zeta(s)$ are all positive (equal to 1). Does not extend to $L(1/2 + it, \chi_{-4})$ for example.

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Theorem: (de la Breteche & Tenenbaum) If q is a large prime, there exist primitive characters $\chi \pmod q$ such that

$$|L(\tfrac{1}{2}, \chi)| \ge \exp\left((1-\epsilon)\frac{\sqrt{\log q \log_3 q}}{\sqrt{\log_2 q}}\right).$$

Don't have a similar result for $L(\frac{1}{2}, \chi_d)$, or for $|L(\frac{1}{2}+i, \chi)|$.

Large values for $1 > \sigma > 1/2$

Refinement pioneered by Aistleitner.

Theorem: (Aistleitner) Fix $1 > \sigma > \frac{1}{2}$. For large T,

$$\max_{\sqrt{T} \le t \le T} |\zeta(\sigma + it)| \ge \exp\left(C(\sigma) \frac{(\log T)^{1-\sigma}}{(\log \log T)^{\sigma}} \right).$$

"Only" recovers result of Montgomery in this situation.

Introduces the possibility of long resonators.

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Theorem: (Aistleitner, Mahatab, Munsch, Peyrot) For large prime q, there exist primitive characters $\chi \pmod{q}$ such that

$$|L(\sigma, \chi)| \ge \exp\left(C(\sigma) \frac{(\log q)^{1-\sigma}}{(\log\log q)^{\sigma}} \right).$$

Open problem: Corresponding result for $L(\sigma, \chi_d)$.

Large values values for $\sigma = 1$

Theorem: (Aistleitner, Mahatab, Munsch) There is a constant C such that

$$\max_{\sqrt{T} \le t \le T} |\zeta(1+it)| \ge e^{\gamma}(\log_2 T + \log_3 T - C).$$

Granville & S. conjecture: for a specified constant C_1

$$\max_{T \le t \le 2T} |\zeta(1+it)| = e^{\gamma}(\log_2 T + \log_3 T + C_1 + o(1)).$$

Established a slightly weaker lower bound — off by $\log_4 T$.

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Theorem: (Aistleitner, Mahatab, Munsch, Peyrot) If q is a large prime there is a primitive character χ with

$$|L(1,\chi)| \ge e^{\gamma}(\log_2 q + \log_3 q - C).$$

Large values of $L(\sigma,\chi)$

Large sieve/zero density estimates: Apart from $\le \sqrt{q}$ characters

$$\log L(\sigma,\chi) \approx \sum_{p\le X} \frac{\chi(p)}{p^\sigma}, \quad X = (\log q)^{10}.$$

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Resonator: (biases $\chi(p)$ towards 1 for $p \leq y$)

$$R(\chi) = \prod_{p \leq y} \left(1 - \frac{\chi(p)}{2}\right)^{-1} = \sum_{n \in \mathcal{S}(y)} \frac{\chi(n)}{2^{\Omega(n)}}.$$

Lower bound the ratio I_2/I_1 where

$$I_1 = \sum_{\chi \pmod{q}} |R(\chi)|^2, \quad I_2 = \sum_{\chi \pmod{q}} \left(\sum_{p \leq X} \frac{\chi(p)}{p^{\sigma}}\right) |R(\chi)|^2.$$

If the contribution of $\leq \sqrt{q}$ bad characters to I_2 is negligible, this produces large values of $L(\sigma, \chi)$.

$$
\begin{aligned}
I_2 &= \sum_{\chi \pmod q} \Big(\sum_{p \le X} \frac{\chi(p)}{p^\sigma} \Big) \sum_{m,n \in \mathcal{S}(y)} \frac{\chi(m)}{2^{\Omega(m)}} \frac{\overline{\chi(n)}}{2^{\Omega(n)}} \\
&= \phi(q) \sum_{\substack{pm \equiv n \pmod q \\ m,n \in \mathcal{S}(y)}} \frac{1}{2^{\Omega(m)+\Omega(n)} p^\sigma}.
\end{aligned}
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$$= \phi(q) \sum_{\substack{pm \equiv n \pmod q \\ m,n \in \mathcal{S}(y)}} \frac{1}{2^{\Omega(m)+\Omega(n)} p^\sigma}.$$

All terms are positive! Focus just on the terms where $n = pk$. Assuming $y \le X$, these terms alone give

$$I_2 \ge \phi(q) \sum_{p \le y} \frac{1}{p^\sigma} \frac{1}{2} \sum_{\substack{m \equiv k \pmod q \\ m,k \in \mathcal{S}(y)}} \frac{1}{2^{\Omega(m)+\Omega(k)}} \gg \frac{y^{1-\sigma}}{\log y} I_1.$$

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Need: contribution of the $\le \sqrt{q}$ bad characters is negligible. Since $|R(\chi)| \le 2^{\pi(y)}$, this is

$$\le \sqrt{q} 2^{2\pi(y)} \sum_{p \le X} \frac{1}{p^\sigma} \le q^{\frac{3}{4}}, \quad \text{if } y \le \frac{1}{10} \log q \log\log q.$$

Conclusion: With $y = \frac{1}{10}\log q \log\log q$, there are primitive characters $\chi \pmod q$ with

$$\log|L(\sigma,\chi)| \approx \mathrm{Re}\sum_{p\le X}\frac{\chi(p)}{p^\sigma} \gg \frac{y^{1-\sigma}}{\log y} \gg \frac{(\log q)^{1-\sigma}}{(\log\log q)^\sigma}.$$

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Features/Limitations of the method

Need positivity in the orthogonality relations:

$$\frac{1}{\phi(q)} \sum_{\chi \pmod q} \chi(m)\overline{\chi(n)} = \begin{cases} 1 & \text{if } m \equiv n \pmod q \\ 0 & \text{otherwise.} \end{cases}$$

Works also for even characters, but not odd characters:

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi(-1)=-1}} \chi(m)\overline{\chi(n)} = \begin{cases} \pm 1 & \text{if } m \equiv \pm n \pmod q \\ 0 & \text{otherwise} \end{cases}$$

Analogue in t–aspect:

$$\frac{1}{T}\int_{-\infty}^{\infty}\Phi\Big(\frac{t}{T}\Big)\Big(\frac{m}{n}\Big)^{it}dt = \widehat{\Phi}(T\log(n/m)).$$

Can choose smooth Φ with $\widehat{\Phi}(\xi) \geq 0$.
Note: Such Φ will necessarily be supported near 0.
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Reason for $\sqrt{T} \leq t \leq T$ and cannot localize to $[T, 2T]$.
Need positivity of coefficients of L–functions.
E.g. cannot work with

$$\log L(\sigma, f \times \chi) \approx \sum_{p \leq X} \lambda_f(p) \frac{\chi(p)}{p^\sigma}$$

or

$$\log |L(\sigma + i, \chi)| \approx \mathrm{Re} \sum_{p \leq X} p^{-i} \frac{\chi(p)}{p^\sigma}.$$

Resonator exploits terms of much larger size than previously!

$$I_1 = \frac{1}{\phi(q)} \sum_{\chi \pmod q} |R(\chi)|^2 \geq \sum_{n \in \mathcal{S}(y)} \frac{1}{4^{\Omega(n)}} = \left(\frac{4}{3}\right)^{\pi(y)} \geq q^{\theta}.$$

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Contribution of terms smaller than x to the resonator:

$$\sum_{\substack{n \in \mathcal{S}(y) \\ n \leq x}} \frac{1}{2^{\Omega(n)}} \leq x^{\delta} \sum_{n \in \mathcal{S}(y)} \frac{1}{2^{\Omega(n)} n^{\delta}} = x^{\delta} \prod_{p \leq y} \left(1 - \frac{1}{2p^{\delta}}\right)^{-1}$$

For any fixed $\delta > 0$ this is

$$\ll x^{\delta} \exp\left(C_{\delta} \frac{y^{1-\delta}}{\log y}\right) = x^{\delta} q^{o(1)}.$$

Negligible for $x = q^A$ with A arbitrarily large.
Resonator uses terms of size about $q^{\log \log q}$.

Bondarenko & Seip's result on gcd sums

Theorem: (Bondarenko & Seip) There is a set $\mathcal{N} = \{n_1, \ldots, n_N\}$ of N natural numbers, and positive real numbers c_1, …, c_N such that

$$\sum_{k,\ell=1}^{N} c_k c_\ell \frac{(n_k, n_\ell)}{\sqrt{n_k n_\ell}} \geq \Big(\sum_k c_k^2 \Big) \exp\left((1-\epsilon) \frac{\sqrt{\log N \log_3 N}}{\sqrt{\log_2 N}} \right).$$

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$\mathcal{N}$ chosen as a set of square-free numbers that are divisor closed
That is, if $n \in \mathcal{N}$ all $d|n$ are also in $\mathcal{N}$.
Divide $\mathcal{P} = [e \log N \log_2 N, e^K \log N \log_2 N]$ into blocks $\mathcal{P}_k = [e^k \log N \log_2 N, e^{k+1} \log N \log_2 N]$.
Will take $K = \log_2 N$.

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$\mathcal{N}$ square-free numbers containing at most m_k primes from $\mathcal{P}_k$.
c_k will be a multiplicative function $f(n_k)$ – f a modified version of the function in resonance method.
On $\mathcal{P}_k$, take $f(p) = A(k)/\sqrt{p}$ — throughout $\mathcal{P}$,
$1 \geq f(p) \geq 1/\sqrt{p}$.

Restrict just to $n_\ell | n_k$. Since $\mathcal{N}$ is divisor closed:

$$\sum_{k,\ell=1}^{N} c_k c_\ell \frac{(n_k, n_\ell)}{\sqrt{n_k n_\ell}} \geq \sum_{k=1}^{N} \frac{f(n_k)}{\sqrt{n_k}} \sum_{n_\ell | n_k} f(n_\ell)\sqrt{n_\ell}.$$

Trivially

$$\sum_k c_k^2 \leq \sum_n f(n)^2 = \prod_p \Big(1 + f(p)^2\Big).$$

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Goals:

1. Make ratio large:

$$\sum_{n=1}^{\infty} \frac{f(n)}{\sqrt{n}} \sum_{d|n} f(d)\sqrt{d} \Big/ \sum_{n=1}^{\infty} f(n)^2.$$

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2. Check that $|\mathcal{N}| \leq N$.
3. Show that

$$\sum_{n \notin \mathcal{N}} \frac{f(n)}{\sqrt{n}} \sum_{d|n} f(d)\sqrt{d} = o(\text{ full sum}).$$

Goal 1:

$$\sum_n \frac{f(n)}{\sqrt{n}} \sum_{d|n} f(d)\sqrt{d} = \prod_p \left(1 + \frac{f(p)}{\sqrt{p}}\left(1 + f(p)\sqrt{p}\right)\right)$$

$$= \prod_p \left(1 + f(p)^2 + \frac{f(p)}{\sqrt{p}}\right).$$

Since $f(p) \leq 1$ always: ratio amounts to maximizing

$$\exp\left(\sum_p \frac{f(p)}{\sqrt{p}}\right).$$

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Goal 2: Recall:
$\mathcal{P}_k = [e^k \log N \log_2 N, e^{k+1} \log N \log_2 N]$,
$\mathcal{N}$ – numbers with at most m_k primes from this interval for each k.

$$\#\mathcal{N} \leq \prod_{k=1}^{K-1} \Big(\sum_{j \leq m_k} \binom{e^{K+1} \log N}{j}\Big) \leq \exp\Big(\sum_k m_k \Big(k + \log \frac{\log N}{m_k}\Big)\Big).$$

Essentially need:

$$\sum_k k m_k \leq \log N.$$

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Goal 3:

Want:

$$\sum_{n \notin \mathcal{N}} \frac{f(n)}{\sqrt{n}} \sum_{d|n} f(d)\sqrt{d} = o(\text{ full sum}).$$

If $n \notin \mathcal{N}$ then for some $1 \leq k \leq K-1$ we must have that n has more than m_k prime factors from $\mathcal{P}_k$.

Enough to show:

$$\sum_{\substack{p|n \implies p \in \mathcal{P}_k \\ \Omega(n) > m_k}} \frac{f(n)}{\sqrt{n}} \prod_{p|n} \left(1 + f(p)\sqrt{p}\right) = o\Big(\prod_{p \in \mathcal{P}_k} \left(1 + f(p)^2 + \frac{f(p)}{\sqrt{p}}\right)\Big).$$

Usual Taylor series argument: constraint

$$m_k \geq \sum_{p \in \mathcal{P}_k} f(p)^2.$$

Problem: Constraints: Goals 2 & 3

$$m_k = \sum_{p \in \mathcal{P}_k} f(p)^2, \quad \sum_k k m_k \le \log N.$$

Maximize: Goal 1

$$\sum_p \frac{f(p)}{\sqrt{p}}.$$

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Motivates: On $\mathcal{P}_k$ take $f(p) = A(k)/\sqrt{p}$ for suitable $A(k)$.
Then

$$m_k = A(k)^2 \sum_{p \in \mathcal{P}_k} \frac{1}{p} = A(k)^2 \log \frac{\log(e^{k+1} \log N \log_2 N)}{\log(e^k \log N \log_2 N)} \approx \frac{A(k)^2}{k + \log_2 N}.$$

Constraint:

$$\sum_k \frac{kA(k)^2}{k + \log_2 N} \le \log N$$

Maximize:

$$\sum_k \frac{A(k)}{k + \log_2 N}.$$

Should choose $A(k) = A/k$ for $1 \leq k \leq \log_2 N$.

Constraint:

$$A^2 \sum_{k \leq \log_2 N} \frac{1}{k(k + \log_2 N)} \leq \log N; \quad A = \sqrt{\frac{\log N \log_2 N}{\log_3 N}}.$$

Ratio:

$$\exp\Big(\sum_{k \leq \log_2 N} \frac{A}{k(k + \log_2 N)} \Big) \approx \exp\Big(\frac{\sqrt{\log N \log_3 N}}{\sqrt{\log_2 N}} \Big).$$

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Bondarenko–Seip choice:

$$f(p) = \sqrt{\frac{\log N \log_2 N}{\log_3 N}} \frac{1}{\sqrt{p}(\log p - \log_2 N - \log_3 N)}.$$

Application to large values of $|L(\frac{1}{2},\chi)|$

Theorem: (de la Breteche & Tenenbaum) If q is a large prime, there exist primitive characters $\chi \pmod q$ such that

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Apply the Bondarenko–Seip construction with $N = \sqrt{q}$.
Let f be the multiplicative function given there.
Resonator coefficients:

$$r(n) = \Big(\sum_{\substack{k \equiv n \pmod q \\ k \in \mathcal{N}}} f(k)^2\Big)^{\frac{1}{2}}.$$

Resonator:

$$R(\chi) = \sum_{n \leq q} r(n)\chi(n).$$

Want: large value of

$$\Big|\sum_{\chi \pmod q} L(\tfrac{1}{2},\chi)|R(\chi)|^2\Big| \Big/ \sum_{\chi \pmod q} |R(\chi)|^2.$$

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Denominator:

$$\phi(q)\sum_{n<q} r(n)^2 = \phi(q)\sum_{n<q}\Big(\sum_{\substack{k\equiv n \pmod q\\ k\in\mathcal{N}}} f(k)^2\Big) = \phi(q)\sum_{k\in\mathcal{N}} f(k)^2.$$

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Numerator: For $\chi \neq \chi_0$ replace $L(\frac{1}{2},\chi)$ by $\sum_{a\le q}\chi(a)/\sqrt(a)$.

$$\sum_{\chi \pmod q}\sum_{a\le q}\frac{\chi(a)}{\sqrt{a}}\sum_{m,n\le q} r(m)r(n)\chi(m)\overline{\chi(n)} + O\Big(\sqrt{q}\Big|\sum_{n\le q} r(n)\Big|^2\Big)$$

$$= \phi(q)\sum_{\substack{a,m,n\le q\\ am\equiv n \pmod q}}\frac{r(m)r(n)}{\sqrt{a}} + O\Big(\sqrt{q}\Big|\sum_{n\le q} r(n)\Big|^2\Big).$$

Handling the error term: Since $\mathcal{N}$ has only $\sqrt{q}$ elements, $r(n) \neq 0$ only on $\leq \sqrt{q}$ residue classes.

$$\sqrt{q}\Big|\sum_{n\leq q} r(n)\Big|^2 \leq \sqrt{q}\sqrt{q}\sum_{n\leq q} r(n)^2 \ll \text{Denominator}.$$

Handling the error term: Since $\mathcal{N}$ has only $\sqrt{q}$ elements, $r(n) \neq 0$ only on $\leq \sqrt{q}$ residue classes.

$$\sqrt{q}\Big|\sum_{n\leq q} r(n)\Big|^2 \leq \sqrt{q}\sqrt{q}\sum_{n\leq q} r(n)^2 \ll \text{Denominator.}$$

Main term: a, m, n given with $am \equiv n \pmod q$.

$$\begin{aligned} r(m)r(n) &\geq \Big(\sum_{\substack{ak=\ell\\ k,\ell\in\mathcal{N}\\ k\equiv m,\ell\equiv n}} f(k)^2\Big)^{\frac{1}{2}}\Big(\sum_{\substack{ak=\ell\\ k,\ell\in\mathcal{N}\\ k\equiv m,\ell\equiv n}} f(\ell)^2\Big)^{\frac{1}{2}} \\ &\geq \sum_{\substack{ak=\ell\\ k,\ell\in\mathcal{N}\\ k\equiv m,\ell\equiv n}} f(k)f(\ell) \end{aligned}$$

Given a, sum over m and n gives

$$\sum_{\substack{m,n\leq q\\ am\equiv n \pmod q}} r(m)r(n) \geq \sum_{\substack{ak=\ell\\ k,\ell\in\mathcal{N}}} f(k)f(\ell).$$

Conclude: Numerator

$$\geq \phi(q) \sum_{a \leq q} \frac{1}{\sqrt{a}} \sum_{\substack{ak=\ell \\ k,\ell \in \mathcal{N}}} f(k) f(\ell) = \phi(q) \sum_{\ell \in \mathcal{N}} \frac{f(\ell)}{\sqrt{\ell}} \sum_{\substack{ak=\ell \\ a \leq q}} f(k)\sqrt{k}.$$

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If $a \leq q$ is ignored, this is exactly the numerator of the gcd sum theorem!

Can get rid of the $a \leq q$ condition, by applying Rankin's trick.

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To sum up:

$$\Big| \sum_{\chi \pmod q} L(\tfrac{1}{2}, \chi) |R(\chi)|^2 \Big| \geq \phi(q) \sum_{\ell \in \mathcal{N}} \frac{f(\ell)}{\sqrt{\ell}} \sum_{k | \ell} f(k)\sqrt{k}$$

$$\sum_{\chi \pmod q} |R(\chi)|^2 \leq \phi(q) \sum_{n} f(n)^2.$$

$$\max_{\chi} |L(\tfrac{1}{2}, \chi)| \geq \text{Ratio} \approx \exp\left(C \frac{\sqrt{\log q \log_3 q}}{\sqrt{\log_2 q}} \right).$$

Other applications of the resonance method

Milicevic: Large values of Hecke–Maass eigenforms on hyperbolic surfaces.

Other applications of the resonance method

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Large character sums: $\chi \pmod q$ primitive character.
When does one have cancelation in $\sum_{n\le x}\chi(n)$?
Granville & S. For every fixed A, there are characters with

$$\Big|\sum_{n\le(\log q)^A}\chi(n)\Big| \ge \Big(\rho(A)+o(1)\Big)(\log q)^A.$$

On GRH: If $\log x/\log\log q\to\infty$ then $\sum_{n\le x}\chi(n)=o(x)$.
Hough: Resonance method in this context & refined many results.
de la Breteche & Tenenbaum: There exist characters χ with

$$\Big|\sum_{n\le\sqrt{q}}\chi(n)\Big| \ge q^{\frac14}\exp\Big(C\frac{\sqrt{\log q\log_3 q}}{\sqrt{\log_2 q}}\Big).$$