Selberg's Central Limit Theorem

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An application

Is every number a class number?
Is every number the class number of an imaginary quadratic field?

$\mathcal{F}(h) = \#\{d \leq X : -d \text{ fundamental discriminant class number } h\}$

$\mathcal{F}(1) = 9$. Watkins calculated $\mathcal{F}(h)$ for all $h \leq 100$. How does $\mathcal{F}(h)$ behave asymptotically?

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$\mathcal{F}(h) = \#\{d \leq X : -d \text{ fundamental discriminant class number } h\}$

$\mathcal{F}(1) = 9$. Watkins calculated $\mathcal{F}(h)$ for all $h \leq 100$. How does $\mathcal{F}(h)$ behave asymptotically? Class number formula: $d > 6$

$$h(-d) = \sqrt{d}L(1, \chi_{-d})/\pi.$$

Since $L(1, \chi_d)$ is of constant size, about X discriminants get squished into $\sqrt{X}$ class numbers.
Conjecture: $h/\log h \ll \mathcal{F}(h) \ll h \log h$.
Genus theory: If $2^\lambda \| h$ then $-d$ has at most $\lambda + 1$ odd prime factors.
Perhaps

$$\mathcal{F}(h) \asymp \frac{h}{\log h} \sum_{\ell \leq \lambda+1} \frac{2^{\ell-1}(\log\log h)^{\ell-1}}{(\ell-1)!}$$

"Granularity" of distribution of $L(1, \chi_{-d})$: In how small an interval $(\alpha, \alpha+\delta)$ does the distribution mimic the distribution of random Euler products?

To get hold of class numbers, would need to understand distribution in intervals of length $\delta \asymp 1/\sqrt{X}$.

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Theorem: (S; refined error term by Lamzouri)

$$\sum_{h\le H} \mathcal{F}(h) = \frac{3\zeta(2)}{\zeta(3)} H^2 + O\Big(\frac{H^2(\log\log H)^3}{(\log H)}\Big).$$

Idea of proof: Can assume $d \le X = H^2 \log\log H$.

$$\sum_{h\le H} \mathcal{F}(h) = \frac{1}{2\pi i}\int_{(c)} \sum_{d\le X} \frac{1}{h(-d)^s} H^s \frac{ds}{s}.$$

Connect $\sum_{d\le X} h(-d)^{-s}$ to $\mathbb{E}(L(1,\mathbb{X})^{-s})$ using class number formula.

Corollary: $\mathcal{F}(H) \ll H^2(\log\log H)^2/(\log H)$. For almost all discriminants $-d$, the class number is not a power of 2 times a bounded odd number.

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More refined conjectures/understanding of $\mathcal{F}(h)$: Holmin, Jones, Kurlberg, McLeman and Petersen. Related to Cohen–Lenstra heuristics.

Conjecture: For h odd $\mathcal{F}(h) \sim \mathcal{C}c(h)h/\log h$ with

$$\mathcal{C} = 15 \prod_{p>2} \prod_{i=2}^{\infty} \Big(1 - \frac{1}{p^i}\Big), \text{ and } c(h) = \prod_{p^n \| h} \prod_{i=1}^{n} \Big(1 - \frac{1}{p^i}\Big)^{-1}.$$

Theorem: (Lamzouri)

$$\sum_{\substack{h \le H \\ h \text{odd}}} \mathcal{F}(h) = \frac{15}{4} \frac{H^2}{\log H} + O\Big(\frac{H^2(\log\log H)^3}{(\log H)^{3/2}}\Big).$$

Based on understanding $L(1, \chi_{-p})$ as p runs over primes $\equiv 3 \pmod 4$.

Analogous problems for real quadratic fields

$d > 0$ fundamental discriminant; fundamental unit $\epsilon_d = (t_d + u_d\sqrt{d})/2$ (solution to Pell equation $t^2 - du^2 = 4$ with smallest $u > 0$)

$$h(d) = \sqrt{d}\frac{L(1, \chi_d)}{\log \epsilon_d}.$$

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Chowla family: $d = 4m^2 + 1$, $\epsilon_d = 2m + \sqrt{d}$.

Dahl & Lamzouri: Distribution of class numbers in the Chowla family.

Matching $L(1, \chi_d)$ with suitable random Euler product. Probability of $X(p) = \pm 1$ or 0 depends on $p \pmod 4$.

Theorem: (Dahl & Lamzouri) $\mathcal{F}_{ch}(h)$ number Chowla fundamental discriminants with class number h

$$\sum_{h \leq H} \mathcal{F}_{ch}(h) \sim \frac{1}{2G} H \log H$$

where $G = 1 - 1/3^2 + 1/5^2 - 1/7^2 + \ldots$ is the Catalan constant.

Can also arrange by size of ϵ_d rather than size of d.
Prime geodesic theorems — Sarnak, Hooley, ... The lengths of closed geodesics on $SL_2(\mathbb{Z})\backslash\mathbb{H}$ correspond to regulators of real discriminants, with multiplicity equal to the class number.
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Other related questions

1. Distribution of values of $\zeta(1 + it)$, or $L(1, \chi)$ for characters $\chi \pmod q$.

Distribution in the complex plane — work of Lamzouri.

Extreme values of $|\zeta(1 + it)|$ tend to have $\arg(\zeta(1 + it))$ being small.

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2. Distribution of other families of L–functions at the edge of the critical strip.

Duke (Artin L-functions), Luo (symmetric square L-functions), Cogdell & Michel (k-th symmetric power), Liu, Royer, Wu, ...

In general, good bounds for the coefficients of L-functions is not known (Ramanujan conjecture). Can still get good bounds for $L(1)$ — Iwaniec, Molteni, Xiannan Li.

3. Similar "almost periodicity" for values in the critical strip, but away from critical line.

Lamzouri: Distribution of $\zeta(\sigma + it)$ for $1/2 < \sigma < 1$.

"Universality theorems" – e.g. recent work of Lamzouri, Lester & Radziwill.

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4. Maximal character sums: $\chi \pmod q$ primitive character

$$M(\chi) = \max_x \Big| \sum_{n \le x} \chi(n) \Big|.$$

If $\chi(-1) = -1$ then

$$\Big| \sum_{n \le q/2} \chi(n) \Big| = \frac{\sqrt{q}}{\pi} |(2 - \overline{\chi}(2)) L(1, \overline{\chi})|.$$

Montgomery & Vaughan, Granville & S., Goldmakher, ...
Bober, Goldmakher, Granville & Koukoulopoulos
$\#\{\chi \pmod q : \ M(\chi) \ge (e^{\gamma}\sqrt{q}/\pi)\tau\}$ lies between

$$\exp\Big(-C\frac{e^{\tau}}{\tau}\Big) \quad \text{and} \quad \exp\Big(-c\frac{e^{\tau}}{\tau}\Big).$$

Further afield

5. Error term in the prime number theorem.

$$\psi(e^t) - e^t = -\sum_{\rho} \frac{e^{t\rho}}{\rho}.$$

$e^{it\gamma}$ for different $\gamma > 0$ behave like independent random variable.
Monach, Rubinstein & Sarnak, ...

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6. Error term in the circle problem.

$$P(x) = \sum_{n \leq x} r(n) - \pi x = -\frac{x^{1/4}}{\pi} \sum_{n} \frac{r(n)}{n^{3/4}} \cos(2\pi\sqrt{nx} + \pi/4).$$

Heath-Brown; Hughes & Rudnick (in annuli).

Open: distribution of the error term in the sphere problem.

$$R^{-1}\Big(\sum_{n \leq R^2} r_3(n) - \frac{4}{3}\pi R^3 \Big).$$

Note: $r_3(n)$ related to the class number of $-4n$, and so to $L(1, \chi_{-4n})$!

Theorem (Selberg)

For T large, and t in $[T, 2T]$

$$\log\zeta(\tfrac{1}{2}+it) \text{ is distributed like a complex Gaussian}$$

with mean 0 *and variance* $\sim \log\log T$.

Theorem (Selberg)

For T large, and t in $[T, 2T]$

$$\log \zeta(\tfrac{1}{2} + it) \text{ is distributed like a complex Gaussian}$$

with mean 0 *and variance* $\sim \log\log T$.

Given a nice fixed domain $\mathcal{B} \subset \mathbb{C}$

$$\frac{1}{T}\mathrm{meas}\left\{ t \in [T, 2T] : \quad \frac{\log \zeta(\frac{1}{2} + it)}{\sqrt{\log\log T}} \in \mathcal{B} \right\} \sim \frac{1}{\pi} \int_{z \in \mathcal{B}} e^{-|z|^2} dx\, dy.$$

Equivalently Re $\log \zeta(\frac{1}{2} + it)$ and Im $\log \zeta(\frac{1}{2} + it)$ are distributed like independent Gaussians with mean 0 and variance $\sim \frac{1}{2} \log\log T$.

For fixed V

$$\frac{1}{T}\text{meas}\left\{t \in [T, 2T] : \frac{\log|\zeta(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log T}} \geq V\right\} \sim \frac{1}{\sqrt{2\pi}}\int_V^\infty e^{-x^2/2}dx.$$

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Typically values of $|\zeta(\frac{1}{2}+it)|$ are either large or small, but not usually of constant size. E.g.

$$|\zeta(\tfrac{1}{2}+it)| \geq \exp(\epsilon\sqrt{\log\log T}),\ \text{50 \% of the time},$$

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Ramachandra's conjecture:

$$\left\{\zeta(\tfrac{1}{2}+it), t \in \mathbb{R}\right\} \text{ is dense in } \mathbb{C}.$$

$$\frac{1}{T}\mathrm{meas}\left\{t \in [T, 2T] : \frac{\arg(\zeta(\frac{1}{2}+it))}{\sqrt{\frac{1}{2}\log\log T}} \geq V\right\} \sim \frac{1}{\sqrt{2\pi}}\int_V^{\infty} e^{-x^2/2}dx.$$

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$$S(t) = \frac{1}{\pi}\arg\zeta(\tfrac{1}{2}+it)$$

$$\begin{aligned} N(T) &= \#\{\rho = \beta + i\gamma : \quad 0 < \gamma \leq T\} \\ &= \frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{7}{8} + S(T) + O(T^{-1}). \end{aligned}$$

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Fujii's work: In the range $h\log T\to\infty$ but $h\leq 1$,

$$N(t+h)-N(t)-\frac{h}{2\pi}\log\frac{T}{2\pi}$$

is approximately normal with mean 0 and variance $\sim\frac{1}{\pi^2}\log(h\log T)$.

Proof of Selberg's theorem for $\log|\zeta(\frac{1}{2}+it)|$

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Analogue of this proof for $S(t)$?

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Step 1: Move away from the critical line.

Relate $\log|\zeta(1/2+it)|$ to $\log|\zeta(\sigma_0+it)|$ for most values of t.

Here

$$\sigma_0 = \frac{1}{2} + \frac{W}{\log T}.$$

Need $W = o(\sqrt{\log\log T})$. Choice: $W = (\log\log\log T)^4$.

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Step 2: For most values of $t \in [T, 2T]$ show that

$$|\zeta(\sigma_0+it)|\Big|\prod_{p\le X}\Big(1-\frac{1}{p^{\sigma_0+it}}\Big)\Big| \approx 1.$$

Need X larger than $T^{1/W}$. Choice $X = T^{1/(\log\log\log T)^2}$.

From Steps 1 and 2: for most t

$$\begin{aligned}\log|\zeta(\tfrac{1}{2}+it)| &\approx \log|\zeta(\sigma_0+it)| \\ &\approx \operatorname{Re}\sum_{n\leq X}\frac{\Lambda(n)}{\log n}\frac{1}{n^{\sigma_0+it}} \approx \operatorname{Re}\sum_{p\leq X}\frac{1}{p^{\sigma_0+it}}.\end{aligned}$$

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Step 3: Compute moments of the sum over primes:

$$\frac{1}{T}\int_T^{2T}\left(\operatorname{Re}\sum_{p\le X}\frac{1}{p^{\sigma_0+it}}\right)^k dt \sim \mu_k\left(\tfrac{1}{2}\log\log T\right)^{k/2},$$

where μ_k denote the Gaussian moments

$$\mu_k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ 1\cdot 3\cdots(k-1) & \text{if } k \text{ is even.} \end{cases}$$

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Gaussian determined by its moments. Hence Selberg's theorem.

Step 1: Moving away from the critical line

Lemma: If $T \le t \le 2T$ and $\sigma > 1/2$ then

$$\int_{t-1}^{t+1} \Big| \log|\zeta(\tfrac{1}{2}+iy)| - \log|\zeta(\sigma+iy)| \Big| dy \ll (\sigma - 1/2) \log T.$$

Conclude: Apart from a set of measure $O(T/A)$,

$$\log|\zeta(\tfrac{1}{2}+it)| = \log|\zeta(\sigma_0+it)| + O(AW).$$

Good if A is large and $AW = o(\sqrt{\log\log T})$.

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$$G(s) = s(s-1)\pi^{-s/2}\Gamma(s/2), \qquad \xi(s) = G(s)\zeta(s)$$

Stirling's formula:

$$\int_{t-1}^{t+1} \left| \log \frac{G(\sigma+iy)}{G(1/2+iy)} \right| dy \ll (\sigma - 1/2)\log T.$$

Lemma equivalent to:

$$\int_{t-1}^{t+1} \left| \log \frac{|\xi(\frac{1}{2}+iy)|}{|\xi(\sigma+iy)|} \right| dy \ll (\sigma - 1/2)\log T.$$

Recall Hadamard factorization formula:

$$\xi(s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}$$

$$|\xi(s)| = \prod_{\rho} \left|1 - \frac{s}{\rho}\right|$$

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$$\int_{t-1}^{t+1} \left| \log \left| \frac{\xi(\frac{1}{2}+iy)}{\xi(\sigma+iy)} \right| \right| dy \leq \int_{t-1}^{t+1} \sum_{\rho} \left| \log \left| \frac{\frac{1}{2}+iy-\rho}{\sigma+iy-\rho} \right| \right| dy$$

$$= \frac{1}{2} \sum_{\rho} \int_{t-1}^{t+1} \left| \log \frac{(\beta-1/2)^2+(y-\gamma)^2}{(\beta-\sigma)^2+(y-\gamma)^2} \right| dy.$$

If $|t-\gamma| \geq 2$:

$$\int_{t-1}^{t+1} \left| \log \frac{(\beta-1/2)^2+(y-\gamma)^2}{(\beta-\sigma)^2+(y-\gamma)^2} \right| dy \ll \frac{(\sigma-1/2)}{(t-\gamma)^2}.$$

Contribution of these zeros:

$$\sum_{\substack{\rho \\ |t-\gamma|\geq 2}} \frac{(\sigma-1/2)}{(t-\gamma)^2} \ll (\sigma-1/2)\log T.$$

Zeros near t: If $|t-\gamma| \le 2$

$$\int_{t-1}^{t+1} \Big| \log \frac{(\beta-1/2)^2+(y-\gamma)^2}{(\beta-\sigma)^2+(y-\gamma)^2} \Big| dy \le \int_{-\infty}^{\infty} \Big| \log \frac{(\beta-1/2)^2+x^2}{(\beta-\sigma)^2+x^2} \Big| dx$$
$$= 2\pi(\sigma-1/2).$$

Contribution of all these zeros is also $\ll (\sigma-1/2)\log T$.

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$$= 2\pi(\sigma-1/2).$$

Contribution of all these zeros is also $\ll (\sigma-1/2)\log T$.
Conclusion:

$$\int_{t-1}^{t+1} \Big| \log \Big| \frac{\zeta(\frac{1}{2}+iy)}{\zeta(\sigma+iy)} \Big| \Big| dy = \int_{t-1}^{t+1} \Big| \log \Big| \frac{\xi(\frac{1}{2}+iy)}{\xi(\sigma+iy)} \Big| \Big| dy$$
$$+ O((\sigma-1/2)\log T)$$
$$\ll (\sigma-1/2)\log T.$$