Selberg's theorem & Analogues for L-functions

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Recap

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$$\sigma_0 = \frac{1}{2} + \frac{W}{\log T}; \quad W = (\log\log\log T)^4.$$

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Step 2: For most values of $t \in [T, 2T]$ show that

$$|\zeta(\sigma_0+it)| \Big| \prod_{p \le X} \Big(1 - \frac{1}{p^{\sigma_0+it}}\Big) \Big| \approx 1.$$

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Step 3: Compute moments of the sum over primes:

$$\frac{1}{T}\int_T^{2T}\Big(\mathrm{Re}\sum_{p\le X}\frac{1}{p^{\sigma_0+it}}\Big)^k dt \sim \mu_k\Big(\frac{1}{2}\log\log T\Big)^{k/2},$$

where μ_k denote the Gaussian moments.

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With $X = T^{1/(\log\log\log T)^2}$ — small power of T — want

$$\frac{1}{T}\int_T^{2T}\left(\mathrm{Re}\sum_{p\le X}\frac{1}{p^{\sigma_0+it}}\right)^k dt$$

$$= \frac{1}{2^k}\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{1}{T}\int_T^{2T}\Big(\sum_{p\le X}\frac{1}{p^{\sigma_0+it}}\Big)^{\ell}\Big(\sum_{p\le X}\frac{1}{p^{\sigma_0-it}}\Big)^{k-\ell}dt.$$

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Mean values of Dirichlet polynomials:

$$\frac{1}{T}\int_T^{2T}\sum_{m\le M}a(m)m^{-it}\sum_{n\le N}a(n)n^{it}\,dt$$

Smoothed version: smooth Φ approx. indicator function of $[1,2]$

$$\frac{1}{T}\int_{t\in\mathbb{R}}\Phi(t/T)\sum_{m\le M}a(m)m^{-it}\sum_{n\le N}a(n)n^{it}\,dt$$

$$=\sum_{m\le M}\sum_{n\le N}a(m)b(n)\widehat{\Phi}(T\log(m/n)).$$

Understanding

$$\sum_{m\le M}\sum_{n\le N} a(m)b(n)\widehat{\Phi}(T\log(m/n)).$$

Φ smooth implies $\widehat{\Phi}(x)$ decays rapidly as $|x|\to\infty$.

$$T|\log(m/n)| \gg T\frac{|m-n|}{\min(m,n)}$$

So if $\min(M,N)\le T/\log T$ then the terms $m\ne n$ are negligible. Left with "diagonal" contribution

$$\sum_{m=n} a(n)b(n)\widehat{\Phi}(0).$$

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Apply to

$$\Big(\sum_{p\le X}\frac{1}{p^{\sigma_0+it}}\Big)^{\ell}=\sum_{m\le X^{\ell}} a_{\ell}(m)m^{-it},\ \Big(\sum_{p\le X}\frac{1}{p^{\sigma_0-it}}\Big)^{k-\ell}=\sum_{n} a_{k-\ell}(n)n^{it}.$$

Note $a_\ell(m) = 0$ unless m has exactly ℓ prime factors all below X.
If these are all distinct – usual case – then $a_\ell(m) = \ell! m^{-\sigma_0}$.

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If these are all distinct – usual case – then $a_\ell(m) = \ell! m^{-\sigma_0}$.
Only have diagonal terms if $\ell = k - \ell$.
If k odd:

$$\frac{1}{T}\int_T^{2T}\left(\mathrm{Re}\sum_{p\le X}\frac{1}{p^{\sigma_0+it}}\right)^k dt \text{ is negligible.}$$

If k even

$$\frac{1}{T}\int_T^{2T}\left(\mathrm{Re}\sum_{p\le X}\frac{1}{p^{\sigma_0+it}}\right)^k dt \approx \frac{1}{2^k}\binom{k}{k/2}\frac{1}{T}\int_T^{2T}\left|\sum_{p\le X}\frac{1}{p^{\sigma_0+it}}\right|^k dt.$$

Diagonal terms give:

$$\frac{1}{2^k}\binom{k}{k/2}\sum_{n\le X^{k/2}} a_{k/2}(n)^2 \approx \frac{1}{2^k}\binom{k}{k/2}\sum_{p_1,\dots,p_{k/2}\le X}\frac{(k/2)!}{(p_1\cdots p_{k/2})^{2\sigma_0}}$$

$$\approx \frac{1}{2^k}\frac{k!}{(k/2)!}(\log\log T)^{k/2} = \mu_k(\tfrac{1}{2}\log\log T)^{k/2}.$$

Step 2: Connecting $\log|\zeta(\sigma_0+it)|$ to the prime sum

Want to show: for most $t \in [T, 2T]$

$$|\zeta(\sigma_0+it)|\Big|\prod_{p\le X}\Big(1-\frac{1}{p^{\sigma_0+it}}\Big)\Big| \approx 1.$$

Step 2: Connecting $\log|\zeta(\sigma_0 + it)|$ to the prime sum

Want to show: for most $t \in [T, 2T]$

$$|\zeta(\sigma_0 + it)| \Big| \prod_{p \le X} \Big(1 - \frac{1}{p^{\sigma_0 + it}}\Big) \Big| \approx 1.$$

Heuristic calculation

$$\zeta(s) \prod_{p \le X} \Big(1 - \frac{1}{p^s}\Big) - 1 = \sum_{p|n \implies p > X} \frac{1}{n^s}.$$

Might expect – if diagonal contribution is correct –

$$\frac{1}{T} \int_T^{2T} \Big| \zeta(\sigma_0 + it) \prod_{p \le X} \Big(1 - \frac{1}{p^{\sigma_0 + it}}\Big) - 1 \Big|^2 dt \approx \sum_{p|n \implies p > X} \frac{1}{n^{2\sigma_0}}.$$

Heuristic continued:

$$\frac{1}{T}\int_T^{2T}\left|\zeta(\sigma_0+it)\prod_{p\le X}\left(1-\frac{1}{p^{\sigma_0+it}}\right)-1\right|^2 dt \approx \exp\left(\sum_{p>X}\frac{1}{p^{2\sigma_0}}\right)-1$$

$$\approx \sum_{p>X}\frac{1}{p^{1+2W/\log T}} \ll \frac{X^{2W/\log T}}{W(\log X)/\log T}.$$

Small if $X > T^{A/W}$ with A large — Answer is $\ll e^{-2A}/A$.

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Hard to work directly with Euler products.

E.g. there could be points where the Euler product is exponentially large in T.

Slogan: Think in Euler products, work with Dirichlet series.

Step 2A: Most of the time can approximate Euler product by a short Dirichlet series.

$$W = (\log\log\log T)^4, \quad X = T^{1/(\log\log\log T)^2}, \quad Y = T^{1/(\log\log T)^2}$$

Define $a(n) = 1$ if
n has at most $100\log_2 T$ prime factors below Y
and at most $100\log_3 T$ prime factors between Y and X.
Put $a(n) = 0$ otherwise.

$$M(s) = \sum_n a(n)\frac{\mu(n)}{n^s} - \text{ short Dirichlet polynomial of length } \leq T^{\epsilon}$$

Lemma: For typical $t \in [T, 2T]$

$$\prod_{p \leq X}\left(1 - \frac{1}{p^{\sigma_0 + it}}\right) \approx M(\sigma_0 + it).$$

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Lemma: For typical $t \in [T, 2T]$

$$\prod_{p \leq X} \left(1 - \frac{1}{p^{\sigma_0 + it}}\right) \approx M(\sigma_0 + it).$$

Step 2B: Typically $\zeta(\sigma_0 + it) M(\sigma_0 + it) \approx 1$.

Step 2A: Approximating by a short Dirichlet series

$$P_1(s) = \sum_{2\le n\le Y} \frac{\Lambda(n)}{n^s \log n}, \quad P_2(s) = \sum_{Y<n\le X} \frac{\Lambda(n)}{n^s \log n}.$$

Lemma: Outside a set of measure $o(T)$, for $t \in [T, 2T]$

$$|P_1(\sigma_0 + it)| \le \log\log T, \quad |P_2(\sigma_0 + it)| \le \log\log\log T.$$

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Proof: Mean square is easy to compute:

$$\frac{1}{T}\int_T^{2T} |P_1(\sigma_0 + it)|^2 dt \approx \sum_{2 \le n \le Y} \frac{\Lambda(n)^2}{n^{2\sigma_0}(\log n)^2} \approx \sum_{p \le Y} \frac{1}{p^{2\sigma_0}} \le \log\log T$$

$$\frac{1}{T}\int_T^{2T} |P_2(\sigma_0 + it)|^2 dt \approx \sum_{Y < p \le X} \frac{1}{p^{2\sigma_0}} \approx \log \frac{\log X}{\log Y} \ll \log_3 T.$$

Observation: If $|z| \le K$ then

$$\left| e^z - \sum_{0 \le k \le 100K} \frac{z^k}{k!} \right| \le e^{-99K}.$$

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By the Lemma, for most $t \in [T, 2T]$,

$$\exp(-P_1(\sigma_0 + it)) \approx \sum_{k \leq 100 \log\log T} (-1)^k \frac{P_1(\sigma_0 + it)^k}{k!},$$

$$\exp(-P_2(\sigma_0 + it)) \approx \sum_{\ell \leq 100 \log_3 T} (-1)^\ell \frac{P_2(\sigma_0 + it)^\ell}{\ell!}.$$

Thus, for typical t,

$$\prod_{p \leq X} \left(1 - \frac{1}{p^{\sigma_0 + it}}\right) \approx \exp(-P_1(\sigma_0 + it) - P_2(\sigma_0 + it))$$

$$\approx M(\sigma_0 + it).$$

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Recall: need $W = o(\sqrt{\log\log T})$, and $X \ge T^{A/W} \ge T^{10/\sqrt{\log\log T}}$. Also

$$\frac{1}{T}\int_T^{2T} \Big| \sum_{2\le n\le X} \frac{\Lambda(n)}{n^{\sigma_0+it}\log n}\Big|^2 dt \approx \sum_{p\le X} \frac{1}{p^{2\sigma_0}} \approx \log\log T.$$

To approximate

$$\prod_{p\le X}\Big(1-\frac{1}{p^{\sigma_0+it}}\Big) \approx \sum_{k=0}^{K} \frac{(-1)^k}{k!}\Big(\sum_{2\le n\le X} \frac{\Lambda(n)}{n^{\sigma_0+it}\log n}\Big)^k,$$

will need $K \ge \sqrt{\log\log T}$.

But then $X^K \ge T^{10}$ — not a short Dirichlet polynomial.

Motivation – the pure Brun sieve

Toy problem: Count $n \leq x$ with $(n, P(z)) = 1$ where $P(z) = \prod_{p \leq z} p$.

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If z^k is small compared to x, can evaluate this short sum on average over n.
Do we get close to the right answer?

Expected Answer:

$$
\begin{aligned}
x \prod_{p \le z} \left(1 - \frac{1}{p}\right) &\approx x \exp\Big(- \sum_{p \le z} \frac{1}{p}\Big) \\
&= x \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \Big(\sum_{p \le z} \frac{1}{p}\Big)^\ell .
\end{aligned}
$$

Expected Answer:

$$x \prod_{p \le z} \Big(1 - \frac{1}{p}\Big) \approx x \exp\Big(-\sum_{p \le z} \frac{1}{p}\Big)$$
$$= x \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \Big(\sum_{p \le z} \frac{1}{p}\Big)^\ell.$$

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If $k \ge 10 \sum_{p \le z} 1/p$ this is close to the right answer.

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Force $z \le x^{1/(20 \log\log x)}$ — loss of $\log\log x$ factor.

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Force $z\le x^{1/(20\log\log x)}$ — loss of $\log\log x$ factor.
Split into different ranges and iterate: E.g. with
$z_1=x^{1/(40\log\log x)}$, $z_2=x^{1/(40\log_3 x)}$, $k_1=10\log\log x$, and
$k_2=10\log_3 x$:

$$\Big(\sum_{\substack{p|d_1\implies p<z_1\\ \Omega(d_1)\le k_1}}\mu(d_1)\delta(d_1|n)\Big)\Big(\sum_{\substack{p|d_2\implies z_1\le p<z_2\\ \Omega(d_2)\le k_2}}\mu(d_2)\delta(d_2|n)\Big).$$

Step 2B: Typically $\zeta(\sigma_0+it)M(\sigma_0+it)\approx 1$

Want to show

$$\frac{1}{T}\int_T^{2T}|\zeta(\sigma_0+it)M(\sigma_0+it)-1|^2dt=o(1).$$

$$\frac{1}{T}\int_T^{2T}\Big(|\zeta(\sigma_0+it)M(\sigma_0+it)|^2-2\mathrm{Re}\ \zeta(\sigma_0+it)M(\sigma_0+it)+1\Big)dt=o(1).$$

Recall $M(s)=\sum_n a(n)\mu(n)/n^s$ is a short Dirichlet polynomial.

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Cross term:

$$\zeta(s)=\sum_{n\le T}\frac{1}{n^s}-\frac{T^{1-s}}{1-s}+O(T^{-\sigma})\approx\sum_{n\le T}\frac{1}{n^s}.$$

$$\frac{1}{T}\int_T^{2T}\zeta(\sigma_0+it)M(\sigma_0+it)=\sum_{\substack{m\\ n\le T}}\frac{a(m)\mu(m)}{m^{\sigma_0}}\frac{1}{n^{\sigma_0}}\frac{1}{T}\int_T^{2T}(mn)^{-it}dt$$

$$\approx 1.$$

Reduced to showing:

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Lemma: h, k positive integers, $1 \geq \sigma > 1/2$

$$\int_{T}^{2T} \left(\frac{h}{k}\right)^{it} |\zeta(\sigma + it)|^2 dt$$
$$= \int_{T}^{2T} \left(\zeta(2\sigma)\left(\frac{(h,k)^2}{hk}\right)^{\sigma} + \left(\frac{t}{2\pi}\right)^{1-2\sigma} \zeta(2-2\sigma)\left(\frac{(h,k)^2}{hk}\right)^{1-\sigma}\right) dt$$
$$+ O(T^{1-\sigma}\min(h,k)).$$

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$$+ O(T^{1-\sigma}\min(h,k)).$$

Letting $\sigma \to 1/2$:

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^2 \left(\frac{h}{k}\right)^{it} \approx \frac{(h,k)}{\sqrt{hk}} \int_T^{2T} \left(\log\frac{t(h,k)^2}{2\pi hk} + 2\gamma\right) dt.$$

Lemma allows computation of $\int_T^{2T} |\zeta(\sigma_0 + it)M(\sigma_0 + it)|^2 dt$.
Conforms to heuristic regarding why
$\int_T^{2T} |\zeta(\sigma_0 + it)M(\sigma_0 + it) - 1|^2 dt$ is small.
Details in a related situation later.

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Details in a related situation later.

Approximations to $\zeta(s)$:

$$\zeta(s) = \sum_{n\le T} \frac{1}{n^s} - \frac{T^{1-s}}{1-s} + O(T^{-\sigma}) \approx \sum_{n\le T} \frac{1}{n^s}.$$

Already good enough to give second moment:

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^2 \sim T \sum_{n\le T} \frac{1}{n} \sim T\log T.$$

Approximate functional equation: (permits fourth moment)

$$\zeta(s) \approx \sum_{n\le\sqrt{t/2\pi}} \frac{1}{n^s} + \pi^{s-1/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \sum_{n\le\sqrt{t/2\pi}} \frac{1}{n^{1-s}}.$$

Related work on mean values

$$A(s) = \sum_{n \le N} a(n) n^{-s}, \quad a(n) \ll n^{\epsilon}.$$

Evaluate

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^2 |A(\tfrac{1}{2} + it)|^2 dt.$$

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$$A(s) = \sum_{n \le N} a(n) n^{-s}, \quad a(n) \ll n^{\epsilon}.$$

Evaluate

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^2 |A(\tfrac{1}{2} + it)|^2 dt.$$

Balasubramanian, Conrey & Heath-Brown: If $N \le T^{1/2-\epsilon}$

$$\approx \sum_{m,n \le N} \frac{a(m)\overline{a(n)}}{[m,n]} \int_0^T \left(\log \frac{t(m,n)^2}{2\pi mn} + 2\gamma \right) dt.$$

Conjecture: holds for all $N \le T^{1-\epsilon}$. Implies Lindelöf Hypothesis.
Bettin, Chandee & Radziwill: holds for $N \le T^{1/2+1/66-\epsilon}$.
Conrey: For $a(n)$ related to $\mu(n)$, holds for $N \le T^{4/7-\epsilon}$. Key ingredient in 40% of zeros lie on the critical line.
Hughes & Young: Similar mean square involving fourth moment of zeta (smaller N).

Variant of the approximate functional equation

$$\xi(s) = G(s)\zeta(s) = \xi(1-s); \quad G(s) = \pi^{-s/2}s(s-1)\Gamma(s/2)$$

$$I(s) = I(\bar{s}) = \frac{1}{2\pi i}\int_{(c)} \xi(z+s)\xi(z+\bar{s})e^{z^2}\frac{dz}{z}$$

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Move line of integration to the left and use the functional equation

$$\xi(z+s)\xi(z+\bar{s}) = \xi(-z+(1-s))\xi(-z+(1-\bar{s}))$$

$$I(s) = \xi(s)\xi(\bar{s}) + \frac{1}{2\pi i}\int_{(-c)} \xi(-z+(1-s))\xi(-z+(1-\bar{s}))e^{z^2}\frac{dz}{z}$$

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Move line of integration to the left and use the functional equation

$$\xi(z+s)\xi(z+\bar{s}) = \xi(-z+(1-s))\xi(-z+(1-\bar{s}))$$

$$I(s) = \xi(s)\xi(\bar{s}) + \frac{1}{2\pi i}\int_{(-c)} \xi(-z+(1-s))\xi(-z+(1-\bar{s})) e^{z^2}\frac{dz}{z}$$

Conclude:

$$|\zeta(s)|^2 = \frac{1}{|G(s)|^2}(I(s) + I(1-\bar{s}))$$
$$\approx \sum_{ab\leq t/2\pi} \frac{1}{(ab)^{\sigma}}\left(\frac{a}{b}\right)^{it} + \left(\frac{t}{2\pi}\right)^{1-2\sigma} \sum_{ab\leq t/2\pi} \frac{1}{(ab)^{1-\sigma}}\left(\frac{a}{b}\right)^{it}$$

Sketch of lemma

$$\int_T^{2T} |\zeta(\sigma+it)|^2 \left(\frac{h}{k}\right)^{it} dt \approx \sum_{ab \le T/2\pi} \left(\frac{1}{(ab)^\sigma} + \left(\frac{T}{2\pi}\right)^{1-2\sigma} \frac{1}{(ab)^{1-\sigma}} \right) \times \int_T^{2T} \left(\frac{ah}{bk}\right)^{it} dt$$

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Since $\min(a,b) \le \sqrt{T}$, if h, k are not too big (e.g. $\le T^{1/2-\epsilon}$) only diagonal terms $ah = bk$ matter.

Parametrize diagonal terms:

$$a = \frac{k}{(h,k)}n, \quad b = \frac{h}{(h,k)}n, \quad n \le N = \sqrt{\frac{T(h,k)^2}{2\pi hk}}.$$

Gives main terms:

$$T\sum_{n\le N}\left(\frac{1}{n^{2\sigma}} + \left(\frac{T}{2\pi}\right)^{1-2\sigma}\frac{1}{n^{2-2\sigma}}\right) \approx T\left(\zeta(2\sigma) + \left(\frac{T}{2\pi}\right)^{1-2\sigma}\zeta(2-2\sigma)\right).$$

Analogues for L-functions in families

Conjecture (Keating-Snaith)

The logarithm of central values of L-functions in families have a normal distribution with suitable mean and variance.

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Examples:

1. Dirichlet L-functions $\pmod q$ — Unitary family. Here $\log |L(\frac{1}{2}, \chi)|$ is supposed to be normal, with

$$\text{Mean} = 0 \qquad \text{Variance} \sim \frac{1}{2} \log \log q.$$

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$$\text{Mean} = 0 \qquad \text{Variance} \sim \frac{1}{2}\log\log q.$$

Note: $L(\frac{1}{2},\chi)$ is complex valued.
Conjecture implies that almost all $L(\frac{1}{2},\chi)$ are non-zero.
Chowla's conjecture: $L(\frac{1}{2},\chi) \neq 0$ for all Dirichlet characters χ.
Khan & Ngo (2016): With q prime, at least $(3/8 + o(1))\phi(q)$ of the characters $\pmod q$ have $L(\frac{1}{2},\chi) \neq 0$.
Pratt (2018): Averaging also over q, one can get $\geq 50.073\%$ non-vanishing.

2. Quadratic Dirichlet L-functions — Symplectic family. For $|d| \leq X$, fundamental discriminant, $\log L(\frac{1}{2}, \chi_d)$ is normal with

$$\text{Mean} = \tfrac{1}{2} \log\log X \qquad \text{Variance} \sim \log\log X.$$

Note: $L(\frac{1}{2}, \chi_d)$ should be a non-negative real number.
S. $L(\frac{1}{2}, \chi_d) \neq 0$ for proportion $7/8$ of the fundamental discriminants d.
Conrey & S. $L(\sigma, \chi_d) \neq 0$ for all $\sigma \in [0, 1]$ for a proportion $\geq 1/5$ of fundamental discriminants d.

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3. Quadratic twists of an elliptic curve E with sign of the functional equation $+1$ — Orthogonal family. Here $\log L(\frac{1}{2}, E \times \chi_d)$ is normal with

$$\text{Mean} = -\tfrac{1}{2} \log\log X \qquad \text{Variance} \sim \log\log X.$$

Note: $L(\frac{1}{2}, E \times \chi_d) \geq 0$ by Waldspurger.
Lots of progress in special cases via algebraic methods.
For example, when E has full rational two torsion, or if E has a three torsion point.

Heuristics

Why normal with predicted mean & variance?
Example 1: Dirichlet L-functions $\pmod q$.

$$\log |L(\tfrac{1}{2},\chi)| \approx \mathrm{Re} \sum_{n\le x} \frac{\chi(n)\Lambda(n)}{\sqrt{n}\log n}$$
$$= \mathrm{Re} \sum_{p\le x} \frac{\chi(p)}{\sqrt{p}} + \mathrm{Re}\, \frac{1}{2} \sum_{p\le\sqrt{x}} \frac{\chi(p)^2}{p} + O(1).$$

If $x = q^{o(1)}$ can compute many moments of the sum over primes – get Gaussian with mean 0 and variance $\sim \frac{1}{2}\log\log x \approx \frac{1}{2}\log\log q$.
The prime square contribution is typically $O(1)$ – variance is bounded.
Just like $\log|\zeta(\frac{1}{2}+it)|$.
In the other families, the key difference is the contribution of prime squares!

Example 2: Quadratic Dirichlet L–functions.

$$\log L(\tfrac{1}{2}, \chi_d) \approx \sum_{p \le x} \frac{\chi_d(p)}{\sqrt{p}} + \frac{1}{2} \sum_{p \le \sqrt{x}} \frac{\chi_d(p^2)}{p}.$$

Note: $\chi_d(p^2) = 1$ and so the contribution of these terms is $\sim \frac{1}{2} \log\log x \sim \frac{1}{2} \log\log |d|$ — this accounts for the mean.

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Note: $\chi_d(p^2) = 1$ and so the contribution of these terms is $\sim \frac{1}{2} \log\log x \sim \frac{1}{2} \log\log |d|$ — this accounts for the mean.
Sum over primes is real — normal with mean 0 and variance $\sim \log\log |d|$.

$$\sum_{|d| \le X}^{\flat} \Big(\sum_{p \le x} \frac{\chi_d(p)}{\sqrt{p}} \Big)^k = \sum_{p_1, \ldots, p_k \le x} \frac{1}{\sqrt{p_1 \cdots p_k}} \sum_{|d| \le X}^{\flat} \Big(\frac{d}{p_1 \cdots p_k} \Big)$$

Terms $p_1 \cdots p_k \neq \square$ cancel out.
Diagonal terms only when k even, and the primes pair up.
Variance:

$$\sum_{p \le x} \frac{1}{p} \sim \log\log X.$$

Example 3: Quadratic twists of an elliptic curve.
E an elliptic curve over $\mathbb{Q}$ of conductor N.
E_d — quadratic twist by fund. disc. d with $(d, 2N) = 1$.
$\mathcal{E}$ — set of fund. disc. for which E_d has root number 1.
If ϵ_E is the root number for E then $\epsilon_E(d) = \epsilon_E \chi_d(-N)$.

$$L(s, E) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-1}$$

Normalization: $|\alpha_p| = |\beta_p| = 1$, $\alpha_p + \beta_p = a(p)$, $\alpha_p \beta_p = 1$.
Functional equation: $s \to 1 - s$.

$$L(s, E_d) = \sum_{n=1}^{\infty} \frac{a(n)\chi_d(n)}{n^s}.$$

Waldspurger's theorem implies

$$L(\tfrac{1}{2}, E_d) \geq 0.$$

$$\log L(\tfrac{1}{2}, E_d) \approx \sum_{n \le x} \frac{\Lambda_E(n)}{\sqrt{n}\log n}\chi_d(n).$$

$$\Lambda_E(n) = \begin{cases} (\alpha_p^k + \beta_p^k)\log p & \text{if } n = p^k \\ 0 & \text{otherwise.} \end{cases}$$

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$$\sideset{}{^\flat}\sum_{\substack{|d| \le X \\ d \in \mathcal{E}}} \Big(\sum_{p \le x} \frac{a(p)}{\sqrt{p}} \chi_d(p) \Big)^k = \sum_{p_1, \ldots, p_k \le x} \frac{a(p_1) \cdots a(p_k)}{\sqrt{p_1 \cdots p_k}} \sideset{}{^\flat}\sum_{\substack{|d| \le X \\ d \in \mathcal{E}}} \Big(\frac{d}{p_1 \cdots p_k} \Big)$$

If x^k small compared to X, only terms with $p_1 \cdots p_k = \square$ matter.
Main term only if k even, and the primes pair up:

$$\sim \#\{d\} \frac{k!}{2^{k/2}(k/2)!} \Big(\sum_{p \le x} \frac{a(p)^2}{p} \Big)^{k/2}.$$

Sum over primes is Gaussian with mean 0 and variance

$$\sum_{p \leq x} \frac{a(p)^2}{p} \sim \log\log x \sim \log\log X$$

Rankin–Selberg theory.

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Rankin–Selberg theory. Contribution of squares of primes:

$$\begin{aligned} \sum_{p \leq \sqrt{x}} \frac{\Lambda_E(p^2)}{p\log(p^2)} &= \frac{1}{2} \sum_{p \leq \sqrt{x}} \frac{\alpha_p^2 + \beta_p^2}{p} \\ &= \frac{1}{2} \sum_{p \leq \sqrt{x}} \frac{(\alpha_p + \beta_p)^2 - 2}{p} \sim -\frac{1}{2}\log\log X. \end{aligned}$$

Sum over primes is Gaussian with mean 0 and variance

$$\sum_{p\le x} \frac{a(p)^2}{p} \sim \log\log x \sim \log\log X$$

Rankin–Selberg theory. Contribution of squares of primes:

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Conclude:

$$\log L(\tfrac{1}{2}, E_d) \approx \sum_{p\le x} \frac{a(p)}{\sqrt{p}} + \frac{1}{2}\sum_{p\le\sqrt{x}} \frac{\alpha_p^2+\beta_p^2}{p}$$

is Gaussian with mean $-\frac{1}{2}\log\log X$ and variance $\log\log X$.

Progress towards the conjectured Central Limit Theorems

Idea: Zeros of L-functions near $\frac{1}{2}$ should make the central value small. So one might hope for upper bounds on the frequency with which

$$\frac{\log L(\frac{1}{2}) - \text{Mean}}{\sqrt{\text{Variance}}} \geq V.$$

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Assuming GRH, a version of this idea with attention to uniformity in V leads to sharp upper bounds for moments in families. (S., plus sharp refinement by Adam Harper — to be explained)

Hough: version of such an upper bound (for V of constant size), assuming suitable zero density theorems. E.g.

$$\#\{|d| \leq X : \log|L(\tfrac{1}{2}, \chi_d)| - \tfrac{1}{2}\log\log X \geq V\sqrt{\log\log X}\}|$$

is at most

$$\#\{|d| \leq X\}\Big(\frac{1}{\sqrt{2\pi}}\int_V^\infty e^{-u^2/2}du + o(1)\Big).$$

Upper bound principle

Whenever one can compute some moment (plus epsilon) in a family of L–functions, then one can obtain a one-sided CLT as above. (Radziwill & S., 2014)

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Example: Quadratic twists of an elliptic curve E over $\mathbb{Q}$ of conductor N.

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In this family asymptotics are known only for the first moment:

$$\sum_{\substack{|d| \leq X \\ d \in \mathcal{E}}} L(\tfrac{1}{2}, E_d) \sim C(1, E)X.$$

On GRH one can prove the second moment: (S. & Young)

$$\sum_{\substack{|d| \leq X \\ d \in \mathcal{E}}} L(\tfrac{1}{2}, E_d)^2 \sim C(2, E)X \log X.$$

Theorem: (Radziwill & S) Let V be a fixed real number. For large X we have

$$\left|\left\{d \in \mathcal{E},\ 20 < |d| \le X :\ \frac{\log L(\frac{1}{2}, E_d) + \frac{1}{2}\log\log|d|}{\sqrt{\log\log|d|}} \ge V\right\}\right|$$

is at most

$$|\{d \in \mathcal{E}, |d| \le X\}|\Big(\frac{1}{\sqrt{2\pi}}\int_V^\infty e^{-\frac{x^2}{2}}dx + o(1)\Big).$$

Corollary The values $L(\frac{1}{2}, E_d)$ tend to be small. For all but $o(X)$ fundamental discriminants $|d| \le X$, $d \in \mathcal{E}$,

$$L(\tfrac{1}{2}, E_d) \le (\log X)^{-\frac{1}{2}+\epsilon}.$$

Implication for Tate-Shafarevich groups

Define

$$S(E_d) = L(\tfrac{1}{2}, E_d)\frac{|E_d(\mathbb{Q})_{\mathrm{tors}}|^2}{\Omega(E_d)\mathrm{Tam}(E_d)}.$$

Here:

- $|E_d(\mathbb{Q})_{\mathrm{tors}}|^2$ is bounded.
- $\Omega(E_d)$ is the real period $\asymp 1/\sqrt{|d|}$;
- $\mathrm{Tam}(E_d) = \prod_p T_p(d)$ – Tamagawa numbers; for a generic prime p $T_p(d) = 1$. If $p|d$ then $T_p(d) = c(p)$ where

$$c(p) = 1 + |\{x : \quad f(x) \equiv 0 \pmod{p}\}| = 1, 2, \text{ or } 4,$$

where E is given in Weierstrass form $y^2 = f(x)$.
Birch & Swinnerton-Dyer: If $L(\frac{1}{2}, E_d) \neq 0$ then $S(E_d)$ is the size of the Tate–Shafarevich group $Ш(E_d)$.

Conjecture: Radziwill & S; Delaunay. $\log(|\text{Ш}(E_d)|/\sqrt{|d|})$ has a normal distribution with mean $\mu(E)\log\log X$ and variance $\sigma(E)^2\log\log X$.
$K =$ splitting field of f over $\mathbb{Q}$, $G = \mathrm{Gal}(K/\mathbb{Q})$.
View G as a subgroup of S_3 and let $c(g) = 1+$ number of fixed points of g.

$$\mu(E) = -\frac{1}{2} - \frac{1}{|G|}\sum_{g\in G}\log c(g); \quad \sigma(E)^2 = 1 + \frac{1}{|G|}\sum_{g\in G}(\log c(g))^2.$$

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One can give $\mu(E)$ and $\sigma(E)^2$ explicitly:

$$K = \mathbb{Q}, \qquad \mu(E) = -\tfrac{1}{2} - 2\log 2, \quad \sigma(E)^2 = 1 + 4(\log 2)^2.$$

$$[K:\mathbb{Q}] = 2, \quad \mu(E) = -\tfrac{1}{2} - \tfrac{3}{2}\log 2, \quad \sigma(E)^2 = 1 + \tfrac{5}{2}(\log 2)^2.$$

$$[K:\mathbb{Q} = 3, \quad \mu(E) = -\tfrac{1}{2} - \tfrac{2}{3}\log 2, \quad \sigma(E)^2 = 1 + \tfrac{4}{3}(\log 2)^2.$$

$$[K:\mathbb{Q}] = 6, \quad \mu(E) = -\tfrac{1}{2} - \tfrac{5}{6}\log 2, \quad \sigma(E)^2 = 1 + \tfrac{7}{6}(\log 2)^2.$$

Theorem: Radziwill & S. For fixed $V \in \mathbb{R}$ and as $X \to \infty$,

$$\left|\left\{d \in \mathcal{E},\ 20 < |d| \le X : \frac{\log(S(E_d)/\sqrt{|d|}) - \mu(E)\log\log|d|}{\sqrt{\sigma(E)^2 \log\log|d|}} \ge V\right\}\right|$$

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$$|\{d \in \mathcal{E}, |d| \leq X\}|\left(\frac{1}{\sqrt{2\pi}}\int_V^\infty e^{-\frac{x^2}{2}}\,dx + o(1)\right).$$

Idea:

$$\log \mathrm{Tam}(d) = O(1) + \sum_{p|d} \log c(p).$$

Additive function, and an Erdős-Kac type theorem applies.
Need a little care to make sure that this normal distribution does not interfere with the normal distribution of $\log L(\frac{1}{2}, E_d)$, but relatively standard.
Note: the sum of two independent normal distributions is normal.

What about lower bounds?

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1. Algebraic. (Shimura), Rohrlich, Chinta ... .

If the L-values in the family are Galois conjugate, then showing one of them is non-zero is enough to show all are.

E.g. Chinta: $L(\frac{1}{2}, E \times \chi)$, as χ ranges over all characters $\pmod{p}$ with suitably large order.

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2. On GRH, compute 1-level density of low-lying zeros.

Özluk-Snyder, Brumer, Heath-Brown, Katz-Sarnak, Iwaniec-Luo-Sarnak, ...

E.g. Heath-Brown: For $\geq 1/4$-ths of $d \in \mathcal{E}$ the central value $L(\frac{1}{2}, E_d) \neq 0$.

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3. Mollifier method. Need to compute two moments with a little bit to spare. Selberg, Levinson, Conrey, Kowalski-Michel-Vanderkam, Iwaniec-Sarnak, Khan–Ngo, Pratt, ...

Can refine methods 2 and 3 to obtain L-values of typical size.

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Theorem: Radziwiłł & S., in progress Assume GRH.

$$\#\left\{|d| \le X : d \in \mathcal{E}, \frac{\log L(\frac{1}{2}, E_d) + \frac{1}{2}\log\log X}{\sqrt{\log\log X}} \in (\alpha, \beta)\right\}$$

is at least

$$\#\{|d| \le X : d \in \mathcal{E}\}\left(\frac{1}{4}\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-u^2/2}du + o(1)\right).$$

S./Hough: GRH + Katz-Sarnak conjectures for 1-level density of zeros in families imply Keating-Snaith CLT conjectures.

If one can access the mollifier method, can get lower bounds unconditionally.

Theorem: Radziwiłł & S., in progress

$$\#\Big\{|d| \le X : \frac{\log|L(\frac{1}{2},\chi_d)| - \frac{1}{2}\log\log X}{\sqrt{\log\log X}} \in (\alpha,\beta)\Big\}$$

is at least

$$\#\{|d| \le X\}\Big(\frac{7}{8}\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-u^2/2}du + o(1)\Big).$$

Ideas behind one sided CLT

Want to upper bound

$$\#\{|d| \leq X, d \in \mathcal{E}, \log L(\tfrac{1}{2}, E_d) + \frac{1}{2}\log\log X \geq V\sqrt{\log\log X}\}.$$

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$$\#\{|d| \leq X, d \in \mathcal{E}, \log L(\tfrac{1}{2}, E_d) + \frac{1}{2} \log \log X \geq V \sqrt{\log \log X}\}.$$

$$\mathcal{P}(d) = \sum_{p \leq z} \frac{a(p)}{\sqrt{p}} \chi_d(p), \quad z = X^{1/(\log \log X)^2}$$

Three possibilities:

1. $\mathcal{P}(d) \geq (V - \epsilon)\sqrt{\log \log X}$.
2. $|\mathcal{P}(d)| \geq \log \log X$.
3. $|\mathcal{P}(d)| \leq \log \log X$, but

$$L(\tfrac{1}{2}, E_d)(\log X)^{\frac{1}{2}} \exp(-\mathcal{P}(d)) \geq \exp(\epsilon \sqrt{\log \log X}).$$

Goal: Cases 2 and 3 are rare. Case 1 happens with Gaussian probability.

Handling Cases 1 & 2

$z = X^{1/(\log\log X)^2}$ small — can compute any fixed moment of $\mathcal{P}(d)$

$$\sideset{}{^\flat}\sum_{\substack{|d|\le X\\ d\in\mathcal{E}}} \Big(\sum_{p\le z} \frac{a(p)}{\sqrt{p}} \chi_d(p) \Big)^k = \sum_{p_1,\ldots,p_k\le z} \frac{a(p_1)\cdots a(p_k)}{\sqrt{p_1\cdots p_k}} \sum_{\substack{|d|\le X\\ d\in\mathcal{E}}} \Big(\frac{d}{p_1\cdots p_k} \Big)$$

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Nuisance: d must be square-free; split into progressions mod N to keep track of root number.

Only diagonal terms $p_1\cdots p_k = \square$ matter. Give (roughly)

$$\#\{d\}\sum_{\substack{p_1,\ldots,p_k\le z\\ p_1\cdots p_k=\square}}\frac{a(p_1)\cdots a(p_k)}{\sqrt{p_1\cdots p_k}}.$$

Generic situation: $q_1 < q_2 < \ldots < q_{k/2}$ distinct primes each appearing twice

$$\#\{d\}\binom{k}{2}\binom{k-2}{2}\cdots\binom{2}{2}\sum_{q_1<\ldots<q_{k/2}\le z}\frac{a(q_1)^2\cdots a(q_{k/2})^2}{q_1\cdots q_{k/2}}$$

Conclude: $\mathcal{P}(d)$ has Gaussian moments. Odd moments are small. For k even

$$\begin{aligned}
\sideset{}{^\flat}\sum_{\substack{|d|\le X\\ d\in\mathcal{E}}} \mathcal{P}(d)^k &\sim \#\{d\}\frac{k!}{2^{k/2}(k/2)!}\Big(\sum_{p\le z}\frac{a(p)^2}{p}\Big)^{k/2}\\
&\sim \#\{d\}\frac{k!}{2^{k/2}(k/2)!}(\log\log z)^{k/2}\\
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Case 1: $\mathcal{P}(d) \geq (V-\epsilon)\sqrt{\log\log X}$ happens with probability

$$\frac{1}{\sqrt{2\pi}}\int_{V-\epsilon}^{\infty} e^{-x^2/2}dx \approx \frac{1}{\sqrt{2\pi}}\int_{V}^{\infty} e^{-x^2/2}dx.$$

Case 2: $|\mathcal{P}(d)| \geq \log\log X$ happens (take $k=2$) for $\ll X/\log\log X$ fundamental discriminants d.

Handling Case 3

There are $o(X)$ fundamental discriminants $|d| \leq X$, $d \in \mathcal{E}$ with $|\mathcal{P}(d)| \leq \log\log X$ but

$$L(\tfrac{1}{2}, E_d)(\log X)^{\frac{1}{2}} \exp(-\mathcal{P}(d)) \geq \exp(\epsilon\sqrt{\log\log X}).$$

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Idea: Use truncated Taylor series to replace "Euler product" $\exp(-\mathcal{P}(d))$ by short Dirichlet polynomial.

Lemma: $E_\ell(x) = \sum_{j=0}^{\ell} x^j/j!$. Suppose ℓ is even, and $x \leq \ell/e^2$. Then

$$e^x \leq \left(1 + \frac{e^{-\ell}}{16}\right) E_\ell(x).$$

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Proof: Exercise: If $x < 0$ then $e^x \le E_\ell(x)$.
If $0 \le x \le \ell/e^2$, then

$$e^x - E_\ell(x) = \sum_{j=\ell+1}^{\infty} \frac{x^j}{j!} \le \frac{x^\ell}{\ell!} \sum_{j=\ell+1}^{\infty} \left(\frac{x}{\ell}\right)^{\ell-j} \le \frac{1}{6}\frac{x^\ell}{\ell!} \le \frac{e^{-\ell}}{16}.$$

Proposition: Take $\ell = 2\lfloor 10 \log\log X \rfloor$. Then

$$\sum_{\substack{|d| \le X \\ d \in \mathcal{E}}}^{\flat} L(\tfrac{1}{2}, E_d)(\log X)^{\frac{1}{2}} E_\ell(-\mathcal{P}(d)) \ll X \log\log X.$$

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Note $L(\frac{1}{2}, E_d) \ge 0$ always; and $E_\ell(-\mathcal{P}(d)) \ge 0$ always.
Further, if $|\mathcal{P}(d)| \le \log\log X$ then $E_\ell(-\mathcal{P}(d)) \ge \exp(-\mathcal{P}(d))/2$.
Conclude:

$$\#\{d \text{ in Case } 3\} \ll (X \log\log X)/\exp(\epsilon\sqrt{\log\log X}) = o(X).$$

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Advantage of

$$E_\ell(-\mathcal{P}(d)) = \sum_{j=0}^{\ell} \frac{(-1)^j}{j!}\mathcal{P}(d)^j$$

— short Dirichlet polynomial of length $\le z^\ell \le X^{20/\log\log X}$.

Sketch of Proposition

Key step: Write $u = u_1 u_2^2$ with u_1 square-free.

$$\sideset{}{^\flat}\sum_{\substack{|d|\le X \\ d\in\mathcal{E}}} \chi_d(u) L(\tfrac{1}{2}, E_d) = CX\frac{a(u_1)}{\sqrt{u_1}} + O(X^{7/8+\epsilon}u^{3/8}).$$

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Approximate functional equation:

$$L(\tfrac{1}{2}, E_d) \approx 2\sum_{n\le X}\frac{a(n)}{\sqrt{n}}\chi_d(n).$$

Need

$$2\sum_{n\le X}\frac{a(n)}{\sqrt{n}}\sideset{}{^\flat}\sum_{\substack{|d|\le X \\ d\in\mathcal{E}}}\chi_d(nu).$$

Contribution from terms when $nu = \square$.

For $nu \ne \square$, Poisson summation (e.g. Polya–Vinogradov)

$$\sum_{\substack{|d|\le X \\ d\in\mathcal{E}}}\chi_d(nu) \longleftrightarrow \frac{X}{\sqrt{nu}}\sum_{|k|\le nu/X}\left(\frac{k}{nu}\right).$$

Poisson flip is useful if $nu \leq X^2$.

Can comfortably compute first moment, with room to put in short Dirichlet polynomial.

Barely not enough to do the second moment of $L(\frac{1}{2}, E_d)$.

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From $nu = \square$ terms (so $n = u_1 m^2$):

$$\sum_{\substack{|d| \leq X \\ d \in \mathcal{E}}}^{\flat} \chi_d(u) L(\tfrac{1}{2}, E_d) = 2\#\{d\} \sum_{\substack{n \leq X \\ nu = \square}} \frac{a(n)}{\sqrt{n}}$$

$$= 2\#\{d\} \sum_{m \leq \sqrt{X/u_1}} \frac{a(u_1 m^2)}{\sqrt{u_1} m} \sim CX \frac{a(u_1)}{\sqrt{u_1}}.$$

$C = C(E)$ related to $L(1, \mathrm{sym}^2 E)$

$$L(s, \mathrm{sym}^2 E) = \prod_p \left(1 - \frac{\alpha_p^2}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{\beta_p^2}{p^s}\right)^{-1} = \zeta(2s) \sum_{n=1}^{\infty} \frac{a(n^2)}{n^s}.$$

Proposition: Take $\ell = 2\lfloor 10 \log\log X \rfloor$. Then

$$\sideset{}{^\flat}\sum_{\substack{|d|\le X\\ d\in\mathcal{E}}} L(\tfrac{1}{2}, E_d)(\log X)^{\frac{1}{2}} E_\ell(-\mathcal{P}(d)) \ll X \log\log X.$$

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Expand

$$\frac{\mathcal{P}(d)^j}{j!} = \sum_{\substack{p|n \implies p \le z \\ \Omega(n) = j}} \frac{\widetilde{a}(n)}{\sqrt{n}w(n)} \chi_d(n)$$

where $\widetilde{a}$ completely multiplicative with $\widetilde{a}(p) = a(p)$;
$w(n)$ multiplicative with $w(p^\alpha) = \alpha!$.

$$E_\ell(-\mathcal{P}(d)) = \sum_{\substack{p|u \implies p \le z \\ \Omega(u) \le \ell}} (-1)^{\Omega(u)} \frac{\widetilde{a}(u)}{\sqrt{u}w(u)} \chi_d(u).$$

Appeal to Lemma: $u = u_1 u_2^2$

$$CX(\log X)^{\frac{1}{2}} \sum_{\substack{p|u \implies p \le z \\ \Omega(u) \le \ell}} (-1)^{\Omega(u)} \frac{\widetilde{a}(u)}{\sqrt{u}w(u)} \frac{a(u_1)}{\sqrt{u_1}}.$$

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Ignoring condition that $\Omega(u) \le \ell$:

$$CX(\log X)^{\frac{1}{2}} \prod_{p\le z} \Big(1 - \frac{a(p)}{\sqrt{p}} \frac{a(p)}{\sqrt{p}} + \frac{a(p)^2}{2p} + \dots\Big) \ll X(\log X)^{\frac{1}{2}} (\log z)^{-\frac{1}{2}}$$

which gives $\ll X \log\log X$, as needed.

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which gives $\ll X \log\log X$, as needed.

Rankin's trick: omitted terms give ($e^{\Omega(u)-\ell} \ge 1$ on these terms)

$$\ll \frac{X(\log X)^{\frac{1}{2}}}{e^{\ell}} \prod_{p\le z} \left(1 + e\frac{a(p)^2}{p} + e^2 \frac{a(p)^2}{2p} + \ldots\right) \ll X(\log X)^{-10}.$$

Key ingredients in proof. Need:
Compute moments of short sum over primes.
To evaluate first moment of $L(\frac{1}{2}, E_d)$ times a short Dirichlet polynomial.
Positivity of $L(\frac{1}{2}, E_d)$.

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Analogue for $L(\frac{1}{2}, \chi_d)$.
Don't know positivity of $L(\frac{1}{2}, \chi_d)$.
But can work with the second moment of $L(\frac{1}{2}, \chi_d)$ multiplied by a short Dirichlet polynomial.